

# A PRIORI $hp$ -ESTIMATES FOR DISCONTINUOUS GALERKIN APPROXIMATIONS TO LINEAR HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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**ABSTRACT.** An  $hp$ -discontinuous Galerkin (DG) method is applied to a class of second order linear hyperbolic integro-differential equations. Based on the analysis of an expanded mixed type Ritz-Volterra projection, *a priori*  $hp$ -error estimates in  $L^\infty(L^2)$ -norm of the velocity as well as of the displacement, which are optimal in the discretizing parameter  $h$  and suboptimal in the degree of polynomial  $p$  are derived. For optimal estimates of the displacement in  $L^\infty(L^2)$ -norm with reduced regularity on the exact solution, a variant of Baker's nonstandard energy formulation is developed and analyzed. Results on order of convergence which are similar in spirit to linear elliptic and parabolic problems are established for the semidiscrete case after suitably modifying the numerical fluxes. For the completely discrete scheme, an implicit-in-time procedure is formulated, stability results are derived and *a priori* error estimates are discussed. Finally, numerical experiments on two dimensional domains are conducted which confirm the theoretical results.

**Keywords** - Local discontinuous Galerkin method, linear second order hyperbolic integro-differential equation, nonstandard formulation, semidiscrete and completely discrete schemes, mixed type Ritz-Volterra projection, role of stabilizing parameters,  $hp$ -error estimates, order of convergence, numerical experiments.

## 1. INTRODUCTION

In this paper, we discuss discontinuous Galerkin (DG) methods which include the local discontinuous Galerkin method (LDG) for the following second order linear hyperbolic integro-differential equation:

$$(1.1) \quad u_{tt} - \nabla \cdot \left( A(x) \nabla u + \int_0^t B(x, t, s) \nabla u(s) ds \right) = f(x, t) \quad \text{in } \Omega \times (0, T],$$

$$(1.2) \quad u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T],$$

$$(1.3) \quad u|_{t=0} = u_0 \quad \text{in } \Omega,$$

$$(1.4) \quad u_t|_{t=0} = u_1 \quad \text{in } \Omega,$$

where  $u_{tt} = (\partial^2 u / \partial t^2)$ , and  $f, u_0, u_1$  are given functions. We assume that  $\Omega$  is a bounded convex domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega$ ,  $A(x) = [a_{ij}(x)]$  is a  $2 \times 2$  positive definite matrix such that there exists a positive constant  $\alpha$  with  $(A(x)\xi, \xi) \geq \alpha|\xi|^2$ ,  $0 \neq \xi \in \mathbb{R}^2$  and  $B(x, t, s) = [b_{ij}(x, t, s)]$  is a  $2 \times 2$  matrix. Further, assume that all coefficients of  $A$  and  $B$  are smooth and bounded functions with bounded derivatives in their respective domain of definitions say by a positive constant  $M$ . Such classes of problems and nonlinear version, thereof, arise naturally in many applications, such as, in viscoelasticity, see [19] and reference, therein.

In literature, finite Galerkin methods are applied to hyperbolic integro-differential equations and *a priori*  $h$ -error estimates have been extensively studied for the problem (1.1)-(1.4) by Cannon *et al.* [4], Pani *et al.* [15]-[16], Lin *et al.* [14], Sihna [22], Sinha and Pani [23], Yanik and Fairweather [27].

Of late, there has been a lot of activities in discontinuous Galerkin (DG) methods for approximating solutions of partial differential equations. This is mainly due to their flexibility in local mesh adaptivity and in taking care of nonuniform degrees of approximation of the solution whose smoothness may exhibit wide variation over the computational domain. Like finite volume element methods, these methods are element-wise conservative, but are ideally suited to *hp*-adaptivity. For application of DG methods to elliptic and parabolic problems, we may refer Cockburn *et al.* [8] for review of development of DG methods. One such DG method is the local discontinuous Galerkin (LDG) method which allows for arbitrary meshes with hanging nodes, elements of various shapes and piecewise polynomials of varying degrees. Earlier, Cockburn and Shu [9] have introduced this method for convection-diffusion problems and subsequently, *hp*-version error estimates are derived by Castillo *et al.* [6]. The LDG method was then extended to elliptic problems by Cockburn *et al.* [5], Perugia and Schötzau [18] and Gudi *et al.* [12]. In [5], optimal order of convergence of LDG method applied to a Poisson equation has been established. Subsequently, Perugia and Schötzau [18] have discussed *a priori* *hp*-error estimates for linear elliptic problems and then, Gudi *et al.* [12] have derived *hp*-error estimates for nonlinear elliptic problems. For higher order partial differential equations using LDG method, see [7, 9, 13, 25, 26] and references, therein.

In this paper, *hp*-DG methods which, in particular, include the original LDG scheme, are applied to the problem (1.1)-(1.4). Further, it is observed that if polynomials of degree at least  $p$  are used in all the elements, the rates of convergence in the  $L^\infty(L^2)$ -norm of the displacement  $u$  and its velocity  $\mathbf{q} = \nabla u$  are of order  $p + 1/2$  and  $p$ , respectively, provided the stabilization parameter  $C_{11} = O(1)$  with  $C_{22} = 0$ . When  $C_{11} = O(1/h)$ , it is shown that the order of convergence of  $u$  is  $p + 1$ . Based on expanded mixed type Ritz-Volterra projection as an intermediate solution, optimal estimates are derived. Using a variant of Baker's nonstandard formulation, *a priori* estimates in  $L^\infty(L^2)$  norm for the displacement are established with reduced regularity conditions on the exact solution. All the above results are proved for semi-discrete method. Then, an implicit-in-time method is applied to the semi-discrete scheme to provide a completely discrete method and stability results are proved. Again a use of a modified Baker's argument combined with a more finner analysis to take care of integral term yields *a priori* estimates for the displacement in  $\ell^\infty L^2$ -norm. Finally, some numerical experiments for the LDG method have been performed with different degrees of polynomials and numerical results are presented to support the theoretical results. Our *hp*-estimates are also valid for second order wave equations only by making  $B \equiv 0$ . For applications of other DG methods to wave equations, we may refer to [20]-[21], [10]-[11].

Throughout this paper, we denote  $C$  as a generic positive constant which does not depend on the discretizing parameter  $h$  and degree of polynomial  $p$ , but may vary from time to time.

The article is organized as follows. Section 2 deals with preliminaries and basic results to be used subsequently in the rest of the article. In Section 3, we formulate DG methods. Section 4 is devoted to an extended mixed type Ritz-Volterra projection and related estimates. In Section 5, we discuss *a priori* error estimates for the semidiscrete scheme. Section 6 focuses on the completely discrete scheme based on an implicit method and related error estimates are derived. In Section 7, some numerical experiments are conducted to confirm the theoretical results.

## 2. PRELIMINARIES

Let  $\mathcal{T}_h = \{K_i : 1 \leq i \leq N_h\}$ ,  $0 < h < 1$  be a family of triangulation consisting of shape regular finite elements, which decompose the domain  $\Omega$  into a finite number of simplexes  $K_i$ , where  $K_i$  is either a triangle or rectangle. It is, further, assumed that the family of

triangulations satisfies bounded local variation conditions on mesh size and on polynomial degree. Let  $h_i$  be the diameter of  $K_i$  and  $h = \max\{h_i : 1 \leq i \leq N_h\}$ . We denote the set of interior edges of  $\mathcal{T}_h$  by  $\Gamma_I = \{e_{ij} : e_{ij} = \partial K_i \cap \partial K_j, |e_{ij}| > 0\}$  and boundary edges by  $\Gamma_\partial = \{e_{i\partial} : e_{i\partial} = \partial K_i \cap \partial\Omega, |e_{i\partial}| > 0\}$ , where  $|e_k|$  denotes the one dimensional Euclidean measure. Let  $\Gamma = \Gamma_I \cup \Gamma_\partial$ . Note that our definition of  $e_k$  also includes hanging nodes along each side of the finite elements. On this subdivision  $\mathcal{T}_h$ , we define the following broken Sobolev spaces

$$V = \{v \in L^2(\Omega) : v|_{K_i} \in H^1(K_i), \forall K_i \in \mathcal{T}_h\},$$

and

$$\mathbf{W} = \{\mathbf{w} \in \mathbf{L}^2(\Omega) : \mathbf{w}|_{K_i} \in \mathbf{H}^1(K_i), \forall K_i \in \mathcal{T}_h\},$$

where  $H^1(K_i)$  is the standard Sobolev space of order one defined on  $K_i$ ,  $\mathbf{L}^2(\Omega) = (L^2(\Omega))^2$  and  $\mathbf{H}^1(K_i) = (H^1(K_i))^2$ . The associated broken norm and seminorm on  $V$  are defined, respectively, as

$$\|v\|_{H^1(\mathcal{T}_h)} = \left( \sum_{i=1}^{N_h} \|v\|_{H^1(K_i)}^2 \right)^{\frac{1}{2}} \quad \text{and} \quad |v|_{H^1(\mathcal{T}_h)} = \left( \sum_{i=1}^{N_h} |v|_{H^1(K_i)}^2 \right)^{\frac{1}{2}}.$$

We denote the  $L^2$ -inner product by  $(\cdot, \cdot)$  and the norm by  $\|\cdot\|$ . We also use broken Sobolev spaces

$$H^r(\mathcal{T}_h) = \{v \in L^2(\Omega) : \sum_{i=1}^{N_h} \|v\|_{H^r(K_i)}^2 < \infty\},$$

with norm  $\|v\|_{H^r(\mathcal{T}_h)} = \left( \sum_{i=1}^{N_h} \|v\|_{H^r(K_i)}^2 \right)^{\frac{1}{2}}$ .

Further, we define for a Hilbert space  $X$

$$L^p(0, T; X) = \{\phi : [0, T] \rightarrow X : \int_0^T \|\phi(t)\|_X^p dt < \infty\},$$

with norm for  $1 \leq p < \infty$

$$\|\phi\|_{L^p(0, T; X)} = \left( \int_0^T \|\phi(t)\|_X^p dt \right)^{\frac{1}{p}},$$

and for  $p = \infty$ ,

$$\|\phi\|_{L^\infty(0, T; X)} = \text{ess sup}_{t \in (0, T)} \|\phi(t)\|_X.$$

For notational convenience, we denote  $L^p(0, T; X)$  as  $L^p(X)$ .

Let  $e_k \in \Gamma_I$ , that is  $e_k = \partial K_i \cap \partial K_j$  for some neighboring simplexes  $K_i$  and  $K_j$ . Let  $\boldsymbol{\nu}_i$  and  $\boldsymbol{\nu}_j$  be the outward normals to the boundary  $\partial K_i$  and  $\partial K_j$ , respectively. On  $e_k$ , we now define the jump and average of  $v \in V$  as

$$[[v]] = v|_{K_i} \boldsymbol{\nu}_i + v|_{K_j} \boldsymbol{\nu}_j, \quad \{v\} = \frac{v|_{K_i} + v|_{K_j}}{2},$$

respectively, and for  $\mathbf{w} \in \mathbf{W}$ , the jump and average are defined as

$$[[\mathbf{w}]] = \mathbf{w}|_{K_i} \cdot \boldsymbol{\nu}_i + \mathbf{w}|_{K_j} \cdot \boldsymbol{\nu}_j, \quad \{\mathbf{w}\} = \frac{\mathbf{w}|_{K_i} + \mathbf{w}|_{K_j}}{2}.$$

In case,  $e_k \in \partial\Omega$ , that is, there exists  $K_i$  such that  $e_k = \partial K_i \cap \partial\Omega$ , then set the jump and average for  $v$  as

$$[[v]] = v|_{K_i \cap \partial\Omega} \boldsymbol{\nu}, \quad \{v\} = v|_{K_i \cap \partial\Omega},$$

respectively, and for  $\mathbf{w} \in \mathbf{W}$ , the jump and average are defined respectively by

$$[[\mathbf{w}]] = \mathbf{w}|_{K_i \cap \partial\Omega} \cdot \boldsymbol{\nu}, \quad \{\mathbf{w}\} = \mathbf{w}|_{K_i \cap \partial\Omega},$$

where  $\nu$  is the outward normal to the boundary  $\partial\Omega$ . Let  $P_{p_i}(K_i)$  be the space of polynomials of degree less than or equal to  $p_i$  on each triangle  $K_i \in \mathcal{T}_h$  and  $Q_{p_i}(K_i)$  be the space of polynomials of degree less than or equal to  $p_i$  in each variable which are defined on the rectangles  $K_i \in \mathcal{T}_h$ . The discontinuous finite element spaces are considered as

$$V_h = \{v_h \in L^2(\Omega) : v_h|_{K_i} \in Z_{p_i}(K_i)\},$$

and

$$\mathbf{W}_h = \{\mathbf{w}_h \in \mathbf{L}^2(\Omega) : \mathbf{w}_h|_{K_i} \in \mathbf{Z}_{p_i}(K_i)\},$$

where  $\mathbf{Z}_{p_i}(K_i) = (Z_{p_i}(K_i))^2$ ,  $p_i \geq 1$  and  $Z_{p_i}(K_i)$  is either  $P_{p_i}(K_i)$  or  $Q_{p_i}(K_i)$ . For any  $e_k \in \Gamma_I$ , there are two elements  $K_i$  and  $K_j$  such that  $e_k = \partial K_i \cap \partial K_j$ . We associate  $p_k$  to  $e_k$  where  $p_k = \frac{p_i + p_j}{2}$ . For  $e_k \in \Gamma_\partial$ , since there is one element  $K_i$  such that  $e_k = \partial K_i \cap \partial\Omega$ , we write  $p_k = p_i$ . We also denote  $p = \min_{1 \leq i \leq N_h} p_i$ .

Below, we state a Lemma without proof on the approximation properties of the finite element spaces. For a proof, refer to [2].

**Lemma 2.1.** *For  $\phi \in (H^{r_i}(K_i))^d$ ,  $d = 1, 2$ , there exist a positive constant  $C_A$  depending on  $r_i$ , but independent of  $\phi, p_i$  and  $h_i$  and a sequence  $\phi_{p_i}^h \in (Z_{p_i}(K_i))^d, p_i \geq 1$ , such that*

(i) *for any  $0 \leq l \leq r_i$ ,*

$$\|\phi - \phi_{p_i}^h\|_{(H^l(K_i))^d} \leq C_A \frac{h_i^{\min\{r_i, p_i+1\}-l}}{p_i^{r_i-l}} \|\phi\|_{(H^{r_i}(K_i))^d},$$

(ii) *for  $r_i > l + \frac{1}{2}$ ,*

$$\|\phi - \phi_{p_i}^h\|_{(H^l(e_k))^d} \leq C_A \frac{h_i^{\min\{r_i, p_i+1\}-l-\frac{1}{2}}}{p_i^{r_i-l-\frac{1}{2}}} \|\phi\|_{(H^{r_i}(K_i))^d}.$$

For any  $\phi \in \mathbf{W}$ , we define  $\mathbf{I}_h \phi \in \mathbf{W}_h$  by

$$\mathbf{I}_h \phi|_{K_i} = \phi_{p_i}^h, \quad \text{for } K_i \in \mathcal{T}_h.$$

We observe that  $\mathbf{I}_h$  satisfies the local approximation properties given in Lemma 2.1. In a similar manner, we can also define  $I_h \psi$ , for  $\psi \in V$ .

**Lemma 2.2.** ( $L^2$ -projection  $\Pi_h$ ). *Let  $\psi \in \mathbf{H}^{r_i+1}(K_i)$  and  $\psi_h = \Pi_h \psi \in \mathbf{Z}_{p_i}(K_i)$  be the  $L^2$ -projection of  $\psi$  onto  $\mathbf{Z}_{p_i}(K_i)$ . Then the following approximation property holds:*

$$\|\psi - \psi_h\|_{(L^2(K_i))^2} + \frac{h_i^{\frac{1}{2}}}{p_i} \|\psi - \psi_h\|_{(L^2(\partial K_i))^2} \leq C \frac{h_i^{\min(r_i, p_i)+1}}{p_i^{r_i+1}} \|\psi\|_{\mathbf{H}^{r_i+1}(K_i)}.$$

### 3. DISCONTINUOUS GALERKIN METHOD

In order to formulate DG methods for hyperbolic integro-differential equations (1.1)-(1.4), we now introduce the gradient and flux variables as

$$\mathbf{q} = \nabla u, \quad \boldsymbol{\sigma} = A\mathbf{q} + \int_0^t B(t, s)\mathbf{q} ds,$$

and then rewrite (1.1) as a system of equations:

$$(3.1) \quad \mathbf{q} = \nabla u \quad \text{in } \Omega,$$

$$(3.2) \quad \boldsymbol{\sigma} = A\mathbf{q} + \int_0^t B(t, s)\mathbf{q} ds \quad \text{in } \Omega,$$

$$(3.3) \quad u_{tt} - \nabla \cdot \boldsymbol{\sigma} = f \quad \text{in } \Omega.$$

Then, the DG formulation for (3.1)-(3.3) is to seek an approximate solution  $(u_h, \mathbf{q}_h, \boldsymbol{\sigma}_h) : (0, T] \mapsto Z_p(K) \times \mathbf{Z}_p(K) \times \mathbf{Z}_p(K)$  satisfying for all  $K \in \mathcal{T}_h$ , the following system of equations for all  $(v_h, \mathbf{w}_h, \boldsymbol{\tau}_h) \in Z_p(K) \times \mathbf{Z}_p(K) \times \mathbf{Z}_p(K)$ :

$$(3.4) \quad \int_K \mathbf{q}_h \cdot \mathbf{w}_h dx + \int_K u_h \nabla \cdot \mathbf{w}_h dx - \int_{\partial K} \hat{u} \mathbf{w}_h \cdot \boldsymbol{\nu}_K ds = 0,$$

$$(3.5) \quad \int_K A \mathbf{q}_h \cdot \boldsymbol{\tau}_h dx - \int_K \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}_h dx + \int_0^t \int_K B(t, s) \mathbf{q}_h(s) \cdot \boldsymbol{\tau}_h dx ds = 0,$$

$$(3.6) \quad \int_K u_{htt} v_h dx + \int_K \boldsymbol{\sigma}_h \cdot \nabla v_h dx - \int_{\partial K} \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}_K v_h ds = \int_K f v_h dx.$$

Here, the numerical fluxes  $\hat{u}$  and  $\hat{\boldsymbol{\sigma}}$  are defined on  $e_k \in \Gamma_I$ , see [5], as:

$$(3.7) \quad \hat{u}(u_h, \boldsymbol{\sigma}_h) = \{u_h\} + C_{12} \cdot [u_h] - C_{22} [\boldsymbol{\sigma}_h],$$

$$(3.8) \quad \hat{\boldsymbol{\sigma}}(u_h, \boldsymbol{\sigma}_h) = \{\boldsymbol{\sigma}_h\} - C_{11} [u_h] - C_{12} [\boldsymbol{\sigma}_h],$$

and for  $e_k \in \Gamma_\partial$ , i.e.,  $e_k = \partial K \cap \partial\Omega$  for some  $K \in \mathcal{T}_h$ , then the numerical fluxes are denoted by

$$(3.9) \quad \hat{u} = 0$$

$$(3.10) \quad \hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_h|_K - C_{11} u_h|_K \boldsymbol{\nu}_K,$$

where the parameters  $C_{11}, C_{12} \in \mathbb{R}^2$  and  $C_{22}$  are single valued and are to be chosen appropriately. It is observed that the numerical fluxes are conservative and consistent (cf. [1]).

To complete the DG formulation, sum (3.4)-(3.6) over all elements  $K \in \mathcal{T}_h$  and apply the conservative property and the definition of the numerical fluxes to arrive at the following system of equations for all  $(v_h, \mathbf{w}_h, \boldsymbol{\tau}_h) \in V_h \times \mathbf{W}_h \times \mathbf{W}_h$ :

$$\begin{aligned} \int_\Omega \mathbf{q}_h \cdot \mathbf{w}_h dx + \sum_{i=1}^{N_h} \int_{K_i} u_h \nabla \cdot \mathbf{w}_h dx - \int_{\Gamma_I} (\{u_h\} + C_{12} \cdot [u_h] - C_{22} [\boldsymbol{\sigma}_h]) [\mathbf{w}_h] ds &= 0, \\ \int_\Omega A \mathbf{q}_h \cdot \boldsymbol{\tau}_h dx - \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}_h dx + \int_0^t \int_\Omega B(t, s) \mathbf{q}_h(s) \cdot \boldsymbol{\tau}_h dx ds &= 0, \\ \int_\Omega u_{htt} v_h dx + \sum_{i=1}^{N_h} \int_{K_i} \boldsymbol{\sigma}_h \cdot \nabla v_h dx - \int_{\Gamma} (\{\boldsymbol{\sigma}_h\} - C_{11} [u_h] - C_{12} [\boldsymbol{\sigma}_h]) [v_h] ds &= \int_\Omega f v_h dx. \end{aligned}$$

Note that the LDG method is obtained, when  $C_{22} = 0$ , that is, when the numerical flux  $\hat{u}$  does not depend on  $\boldsymbol{\sigma}_h$ .

To rewrite the above system in a compact form, we define the following bilinear and linear forms:

$\mathcal{A} : \mathbf{W} \times \mathbf{W} \rightarrow \mathbb{R}$  as

$$\mathcal{A}(\mathbf{p}, \mathbf{w}) = \int_\Omega \mathbf{p} \cdot \mathbf{w} dx,$$

$\mathcal{A}_1 : V \times \mathbf{W} \rightarrow \mathbb{R}$  as

$$\begin{aligned} \mathcal{A}_1(v, \mathbf{p}) &= \sum_{i=1}^{N_h} \int_{K_i} \mathbf{p} \cdot \nabla v \, dx - \int_{\Gamma} (\{\mathbf{p}\} - C_{12}[\mathbf{p}]) \llbracket v \rrbracket \, ds, \\ &= - \sum_{i=1}^{N_h} \int_{K_i} v \nabla \cdot \mathbf{p} + \int_{\Gamma_I} (\{\llbracket v \rrbracket\} + C_{12} \cdot \llbracket v \rrbracket) [\mathbf{p}] \, ds, \end{aligned}$$

$\mathcal{A}_2 : \mathbf{W} \times \mathbf{W} \rightarrow \mathbb{R}$  as

$$\mathcal{A}_2(\mathbf{p}, \mathbf{w}) = \int_{\Omega} A(x) \mathbf{p} \cdot \mathbf{w} \, dx,$$

$\mathcal{B} : \mathbf{W} \times \mathbf{W} \rightarrow \mathbb{R}$  as

$$\mathcal{B}(t, s; \mathbf{p}(s), \mathbf{w}) = \int_{\Omega} B(t, s) \mathbf{p}(s) \cdot \mathbf{w} \, dx,$$

$J_1 : \mathbf{W} \times \mathbf{W} \rightarrow \mathbb{R}$  as

$$J_1(\mathbf{p}, \mathbf{w}) = \int_{\Gamma_I} C_{22}[\mathbf{p}][\mathbf{w}] \, ds,$$

and  $J : V \times V \rightarrow \mathbb{R}$  as

$$J(\phi, v) = \int_{\Gamma} C_{11}[\phi][v] \, ds.$$

Hence, the DG formulation for the problem (3.1)-(3.3) in compact form is stated as: find  $(u_h, \mathbf{q}_h, \boldsymbol{\sigma}_h) : (0, T] \rightarrow V_h \times \mathbf{W}_h \times \mathbf{W}_h$  such that

$$(3.11) \quad \mathcal{A}(\mathbf{q}_h, \mathbf{w}_h) - \mathcal{A}_1(u_h, \mathbf{w}_h) + J_1(\boldsymbol{\sigma}_h, \mathbf{w}_h) = 0, \quad \forall \mathbf{w}_h \in \mathbf{W}_h,$$

$$(3.12) \quad \mathcal{A}_2(\mathbf{q}_h, \boldsymbol{\tau}_h) - \mathcal{A}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \int_0^t \mathcal{B}(t, s; \mathbf{q}_h(s), \boldsymbol{\tau}_h) \, ds = 0, \quad \forall \boldsymbol{\tau}_h \in \mathbf{W}_h,$$

$$(3.13) \quad (u_{htt}, v_h) + \mathcal{A}_1(v_h, \boldsymbol{\sigma}_h) + J(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h.$$

Following [17], we are now ready to specify the stabilization parameters. We define the set  $\langle K, K' \rangle$  by

$$(3.14) \quad \langle K, K' \rangle = \begin{cases} \emptyset & \text{if } \text{meas}(\partial K \cap \partial K') = 0, \\ \text{interior of } \partial K \cap \partial K' & \text{otherwise.} \end{cases}$$

Assume that the stabilization parameters  $C_{11}$  and  $C_{22}$  in the definition of numerical fluxes in (3.7) and (3.8) are stated, respectively, as

$$(3.15) \quad C_{11}(\mathbf{x}) = \begin{cases} \zeta \min\{\frac{h_i^\alpha}{p_i^{2\alpha}}, \frac{h_j^\alpha}{p_j^{2\alpha}}\} & \text{if } \mathbf{x} \in \langle K_i, K_j \rangle, \\ \zeta \frac{h_i^\alpha}{p_i^{2\alpha}}; & \text{if } \mathbf{x} \in \partial K_i \cap \partial \Omega, \end{cases}$$

and

$$(3.16) \quad C_{22}(\mathbf{x}) = \begin{cases} \kappa \min\{\frac{h_i^\beta}{p_i^{2\beta}}, \frac{h_j^\beta}{p_j^{2\beta}}\} & \text{if } \mathbf{x} \in \langle K_i, K_j \rangle, \\ \kappa \frac{h_i^\beta}{p_i^{2\beta}} & \text{if } \mathbf{x} \in \partial K_i \cap \partial \Omega, \end{cases}$$

where  $\zeta > 0$ ,  $\kappa \geq 0$ ,  $-1 \leq \alpha \leq 0 \leq \beta \leq 1$  are independent of mesh size and  $|C_{12}|$  is of order one. Our main results will be written in terms of the parameters  $\mu^*$  and  $\mu_*$ ,

$$\mu^* = \max\{-\alpha, \hat{\beta}\}, \quad \mu_* = \min\{-\alpha, \hat{\beta}\},$$

where  $\hat{\beta} = 1$ , if  $\kappa = 0$  and  $\hat{\beta} = \beta$ , otherwise.

For each edge, we define

$$(3.17) \quad \Lambda(\mathbf{x}) := \begin{cases} \min\{\frac{h_i}{p_i^2}, \frac{h_j}{p_j^2}\} & \text{if } \mathbf{x} \in \langle K_i, K_j \rangle, \\ \frac{h_i}{p_i^2} & \text{if } \mathbf{x} \in \partial K_i \cap \partial\Omega, \end{cases}$$

and then, we set

$$(3.18) \quad \chi(\mathbf{x}) := \begin{cases} \Lambda(\mathbf{x}) & \text{if } \kappa = 0, \\ C_{22}(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Note that numerical fluxes  $\hat{u}$  and  $\hat{\sigma}$  are consistent and hence, we arrive at the following system of equations for all  $(v_h, \tau_h, \mathbf{w}_h) \in V_h \times \mathbf{W}_h \times \mathbf{W}_h$ ,

$$(3.19) \quad \mathcal{A}(\mathbf{q} - \mathbf{q}_h, \mathbf{w}_h) - \mathcal{A}_1(u - u_h, \mathbf{w}_h) + J_1(\sigma - \sigma_h, \mathbf{w}_h) = 0,$$

$$(3.20) \quad \mathcal{A}_2(\mathbf{q} - \mathbf{q}_h, \tau_h) - \mathcal{A}(\sigma - \sigma_h, \tau_h) + \int_0^t \mathcal{B}(t, s; (\mathbf{q} - \mathbf{q}_h)(s), \tau_h) ds = 0,$$

$$(3.21) \quad (u_{tt} - u_{htt}, v_h) + \mathcal{A}_1(v_h, \sigma - \sigma_h) + J(u - u_h, v_h) = 0.$$

Below, we state two theorems on convergence of the semidiscrete scheme, whose proof can be found in the end of Section 5.

**Theorem 3.1.** *Let  $(u, \mathbf{q}, \sigma)$  be the solution of (3.1)-(3.3) satisfying  $u \in L^\infty(H^{r+2}(\mathcal{T}_h))$  with  $u_{tt} \in L^1(H^{r+2}(\mathcal{T}_h))$  for  $r \geq 0$ . Further, let  $(u_h, \mathbf{q}_h, \sigma_h) \in V_h \times \mathbf{W}_h \times \mathbf{W}_h$  be the solution of (3.11)-(3.13). If  $u_h(0) = \Pi_h u_0$ ,  $u_{ht}(0) = \Pi_h u_1$ ,  $q_h(0) = \mathbf{I}_h \nabla u_0$  and  $\sigma_h(0) = \mathbf{I}_h(A \nabla u_0)$ , then the following estimates hold:*

$$(3.22) \quad \|u_t - u_{ht}\|_{L^\infty(L^2(\Omega))} \leq C \frac{h^{P+D}}{p^{R+S}} \left( \|u_0\|_{H^{r+2}(\mathcal{T}_h)} + \|u_1\|_{H^{r+1}(\mathcal{T}_h)} + \sum_{j=0}^2 \left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^1(H^{r+2}(\mathcal{T}_h))} \right),$$

and

$$(3.23) \quad \begin{aligned} \|\mathbf{q} - \mathbf{q}_h\|_{L^\infty(L^2(\Omega))} + \|\sigma - \sigma_h\|_{L^\infty(L^2(\Omega))} &\leq C \frac{h^P}{p^R} \left( \|u_0\|_{H^{r+2}(\mathcal{T}_h)} + \|u_1\|_{H^{r+2}(\mathcal{T}_h)} \right. \\ &\quad \left. + \sum_{j=0}^2 \left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^1(H^{r+2}(\mathcal{T}_h))} \right), \end{aligned}$$

where  $P = \min\{r + \frac{1}{2}(1 + \mu_*), p + \frac{1}{2}(1 - \mu^*)\}$ ,  $D = \frac{1}{2}(1 + \mu_*)$ ,  $R = r + \min\{\mu_*, 1 - \mu^*\}$  and  $S = \min\{\frac{1}{2}, \mu_*\}$ .

**Theorem 3.2.** *Let  $\Omega$  be a bounded convex polygon domain in  $\mathbb{R}^2$  and let  $(u, \mathbf{q}, \sigma)$  be the solution of (3.1)-(3.3) satisfying  $u \in L^\infty(H^{r+2}(\mathcal{T}_h))$  with  $u_t \in L^1(H^{r+2}(\mathcal{T}_h))$  for  $r \geq 0$ . Further, let  $(u_h, \mathbf{q}_h, \sigma_h) \in V_h \times \mathbf{W}_h \times \mathbf{W}_h$  be the solution of (3.11)-(3.13) with  $u_h(0) = \Pi_h u_0$ . Then the following estimate holds:*

$$(3.24) \quad \|u - u_h\|_{L^\infty(L^2(\Omega))} \leq C \frac{h^{P+D}}{p^{R+S}} \left( \|u_0\|_{H^{r+2}(\mathcal{T}_h)} + \sum_{j=0}^1 \left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^1(H^{r+2}(\mathcal{T}_h))} \right),$$

where  $P, D, R$  and  $S$  as in Theorem 3.1.

TABLE 1.

Orders of convergence for  $r \geq p$  and  $p \geq 1$ .

$C_{22}$	$C_{11}$	$\ \mathbf{q} - \mathbf{q}_h\ , \ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\ $	$\ u - u_h\ $
$0, O(\frac{h}{p^2})$	$O(1)$	$\frac{h^p}{p^r}$	$\frac{h^{p+\frac{1}{2}}}{p^r}$
$0, O(\frac{h}{p^2})$	$O(\frac{p^2}{h})$	$\frac{h^p}{p^r}$	$\frac{h^{p+1}}{p^{r+\frac{1}{2}}}$
$O(1)$	$O(1)$	$\frac{h^{p+\frac{1}{2}}}{p^r}$	$\frac{h^{p+1}}{p^r}$
$O(1)$	$O(\frac{p^2}{h})$	$\frac{h^p}{p^r}$	$\frac{h^{p+\frac{1}{2}}}{p^r}$

As in [17], we can make similar observations based on the results of the above Theorem 3.1.

#### 4. EXTENDED MIXED RITZ-VOLTERRA PROJECTION AND RELATED ESTIMATES

In this section, we introduce an extended mixed Ritz-Volterra projection for our subsequent use.

Define an extended mixed Ritz-Volterra projection as: Find  $(\tilde{u}_h, \tilde{\mathbf{q}}_h, \tilde{\boldsymbol{\sigma}}_h) : (0, T] \rightarrow V_h \times \mathbf{W}_h \times \mathbf{W}_h$  satisfying

$$(4.1) \quad \mathcal{A}(\mathbf{q} - \tilde{\mathbf{q}}_h, \mathbf{w}_h) - \mathcal{A}_1(u - \tilde{u}_h, \mathbf{w}_h) + J_1(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, \mathbf{w}_h) = 0, \quad \forall \mathbf{w}_h \in \mathbf{W}_h,$$

$$(4.2) \quad \mathcal{A}_2(\mathbf{q} - \tilde{\mathbf{q}}_h, \tau_h) - \mathcal{A}(\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h, \tau_h) + \int_0^t \mathcal{B}(t, s; (\mathbf{q} - \tilde{\mathbf{q}}_h)(s), \tau_h) ds = 0, \quad \forall \tau_h \in \mathbf{W}_h,$$

$$(4.3) \quad \mathcal{A}_1(v_h, \boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h) + J(u - \tilde{u}_h, v_h) = 0, \quad \forall v_h \in V_h.$$

For given  $(u, \mathbf{q}, \boldsymbol{\sigma})$ , it is easy to show the existence of a unique solution  $(\tilde{u}_h, \tilde{\mathbf{q}}_h, \tilde{\boldsymbol{\sigma}}_h)$  to the problem (4.1)-(4.3).

With  $\eta_u := u - \tilde{u}_h$ ,  $\boldsymbol{\eta}_{\mathbf{q}} := \mathbf{q} - \tilde{\mathbf{q}}_h$ , and  $\boldsymbol{\eta}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h$ , we state without proof the error estimates, whose proofs after simple modifications can be found in [17].

**Theorem 4.1.** *Let  $(u, \mathbf{q}, \boldsymbol{\sigma})$  be the solution of (3.1)-(3.3). Further, let  $(\tilde{u}_h, \tilde{\mathbf{q}}_h, \tilde{\boldsymbol{\sigma}}_h)$  be the solution of (4.1)-(4.3). Then, there exists a positive constant  $C$  independent of  $h$  and  $p$  such that for  $l = 0, 1, 2$*

$$\left\| \frac{\partial^l \eta_u}{\partial t^l} \right\| \leq C \frac{h^{(P+D)}}{p^{(R+S)}} \sum_{j=0}^l \left( \left\| \frac{\partial^j u}{\partial t^j} \right\|_{H^{r+2}(\mathcal{T}_h)} + \int_0^t \left\| \frac{\partial^j u}{\partial t^j}(s) \right\|_{H^{r+2}(\mathcal{T}_h)} ds \right),$$

and

$$(4.4) \quad \begin{aligned} & \left\| \frac{\partial^l \boldsymbol{\eta}_{\boldsymbol{\sigma}}}{\partial t^l} \right\| + \left\| \frac{\partial^l \boldsymbol{\eta}_{\mathbf{q}}}{\partial t^l} \right\| + \left( \int_{\Gamma_I} C_{22} \left[ \left\| \frac{\partial^l \boldsymbol{\eta}_{\boldsymbol{\sigma}}}{\partial t^l} \right\|^2 \right] ds \right)^{1/2} + \left( \int_{\Gamma} C_{11} \left[ \left\| \frac{\partial^l \eta_u}{\partial t^l} \right\|^2 \right] ds \right)^{1/2} \\ & \leq C \frac{h^P}{p^R} \sum_{j=0}^l \left( \left\| \frac{\partial^j u}{\partial t^j} \right\|_{H^{r+2}(\mathcal{T}_h)} + \int_0^t \left\| \frac{\partial^j u}{\partial t^j}(s) \right\|_{H^{r+2}(\mathcal{T}_h)} ds \right), \end{aligned}$$



where  $P = \min\{r + \frac{1}{2}(1 + \mu_*) , p + \frac{1}{2}(1 - \mu^*)\}$ ,  $D = \frac{1}{2}(1 + \mu_*)$ ,  $R = r + \min\{\mu_*, 1 - \mu^*\}$  and  $S = \min\{\frac{1}{2}, \mu_*\}$ .

## 5. A PRIORI ERROR ESTIMATES

In this section, we shall derive *a priori* error estimates for the semi-discrete DG scheme. Using the extended mixed Ritz-Volterra projection, we rewrite

$$\begin{aligned} u - u_h &:= (u - \tilde{u}_h) - (u_h - \tilde{u}_h) =: \eta_u - \xi_u, \\ \mathbf{q} - \mathbf{q}_h &:= (\mathbf{q} - \tilde{\mathbf{q}}_h) - (\mathbf{q}_h - \tilde{\mathbf{q}}_h) =: \boldsymbol{\eta}_{\mathbf{q}} - \boldsymbol{\xi}_{\mathbf{q}}, \\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h &:= (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h) - (\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}_h) =: \boldsymbol{\eta}_{\boldsymbol{\sigma}} - \boldsymbol{\xi}_{\boldsymbol{\sigma}}. \end{aligned}$$

Since the estimates of  $\eta_u$ ,  $\boldsymbol{\eta}_{\mathbf{q}}$  and  $\boldsymbol{\eta}_{\boldsymbol{\sigma}}$  are known from Theorem 4.1, it is enough to obtain estimates for  $\xi_u$ ,  $\boldsymbol{\xi}_{\mathbf{q}}$  and  $\boldsymbol{\xi}_{\boldsymbol{\sigma}}$ . Now, from (4.1)-(4.3) and (3.19)-(3.21), we arrive at

$$(5.1) \quad \mathcal{A}(\boldsymbol{\xi}_{\mathbf{q}}, \mathbf{w}_h) - \mathcal{A}_1(\xi_u, \mathbf{w}_h) + J_1(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, \mathbf{w}_h) = 0 \quad \forall \mathbf{w}_h \in \mathbf{W}_h,$$

$$(5.2) \quad \mathcal{A}_2(\boldsymbol{\xi}_{\mathbf{q}}, \boldsymbol{\tau}_h) - \mathcal{A}(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) + \int_0^t \mathcal{B}(t, s; \boldsymbol{\xi}_{\mathbf{q}}(s), \boldsymbol{\tau}_h) ds = 0 \quad \forall \boldsymbol{\tau}_h \in \mathbf{W}_h,$$

$$(5.3) \quad (\xi_{u_{tt}}, v_h) + \mathcal{A}_1(v_h, \boldsymbol{\xi}_{\boldsymbol{\sigma}}) + J(\xi_u, v_h) = (\eta_{u_{tt}}, v_h) \quad \forall v_h \in V_h.$$

Estimates of  $\|\xi_u\|$ ,  $\|\boldsymbol{\xi}_{\mathbf{q}}\|$  and  $\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|$  are given in the following lemma.

**Lemma 5.1.** *There exists a constant  $C$ , independent of  $h$  and  $p$ , such that*

$$(5.4) \quad \begin{aligned} \|\xi_{u_t}\| + \|\boldsymbol{\xi}_{\mathbf{q}}\| + \|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\| &\leq C \left( \|\xi_{u_t}(0)\| + \|\boldsymbol{\xi}_{\mathbf{q}}(0)\| \right. \\ &\quad \left. + J_1(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(0), \boldsymbol{\xi}_{\boldsymbol{\sigma}}(0))^{1/2} + J(\xi_u(0), \xi_u(0))^{1/2} + \int_0^T \|\eta_{u_{tt}}\| ds \right). \end{aligned}$$

*Proof.* We differentiate equation (5.1) with respect to  $t$  and choose  $\mathbf{w}_h = \boldsymbol{\xi}_{\boldsymbol{\sigma}}$  in (5.1),  $\boldsymbol{\tau}_h = \boldsymbol{\xi}_{\mathbf{q}_t}$  in (5.2) and  $v_h = \xi_{u_t}$  in (5.3). By adding, we obtain

$$(5.5) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \left( \|\xi_{u_t}\|^2 + \mathcal{A}_2(\boldsymbol{\xi}_{\mathbf{q}}, \boldsymbol{\xi}_{\mathbf{q}}) + J_1(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, \boldsymbol{\xi}_{\boldsymbol{\sigma}}) + J(\xi_u, \xi_u) \right) \\ = (\eta_{u_{tt}}, \xi_{u_t}) - \int_0^t \mathcal{B}(t, s; \boldsymbol{\xi}_{\mathbf{q}}(s), \boldsymbol{\xi}_{\mathbf{q}_t}) ds. \end{aligned}$$

Next, we write the integral term on the right hand side of (5.5) as

$$(5.6) \quad \begin{aligned} \int_0^t \mathcal{B}(t, s; \boldsymbol{\xi}_{\mathbf{q}}(s), \boldsymbol{\xi}_{\mathbf{q}_t}) ds &= \frac{d}{dt} \int_0^t \mathcal{B}(t, s; \boldsymbol{\xi}_{\mathbf{q}}(s), \boldsymbol{\xi}_{\mathbf{q}}) ds - \mathcal{B}(t, t, \boldsymbol{\xi}_{\mathbf{q}}(t), \boldsymbol{\xi}_{\mathbf{q}}) \\ &\quad - \int_0^t \mathcal{B}_t(t, s; \boldsymbol{\xi}_{\mathbf{q}}(s), \boldsymbol{\xi}_{\mathbf{q}}) ds. \end{aligned}$$

Substitute (5.6) in (5.5) and integrate from 0 to  $t$ . Then using the Cauchy-Schwarz inequality, the boundedness of  $B$ , the positive-definite property of  $A$  and setting

$$\|(\xi_u, \xi_{u_t}, \boldsymbol{\xi}_{\mathbf{q}}, \boldsymbol{\xi}_{\boldsymbol{\sigma}})(t)\|^2 = \|\xi_{u_t}(t)\|^2 + \|A^{1/2} \boldsymbol{\xi}_{\mathbf{q}}(t)\|^2 + J_1(\boldsymbol{\xi}_{\boldsymbol{\sigma}}(t), \boldsymbol{\xi}_{\boldsymbol{\sigma}}(t)) + J(\xi_u(t), \xi_u(t)),$$

we arrive at

$$(5.7) \quad \begin{aligned} \|(\xi_u, \xi_{u_t}, \boldsymbol{\xi}_{\mathbf{q}}, \boldsymbol{\xi}_{\boldsymbol{\sigma}})(t)\|^2 &\leq \|(\xi_u, \xi_{u_t}, \boldsymbol{\xi}_{\mathbf{q}}, \boldsymbol{\xi}_{\boldsymbol{\sigma}})(0)\|^2 + 2 \int_0^t \|\eta_{u_{tt}}\| \|\xi_{u_t}\| ds \\ &\quad + C(M, \alpha, T) \left( \int_0^t \|A^{1/2} \boldsymbol{\xi}_{\mathbf{q}}(s)\| \|A^{1/2} \boldsymbol{\xi}_{\mathbf{q}}(t)\| ds \right. \\ &\quad \left. + \int_0^t \|A^{1/2} \boldsymbol{\xi}_{\mathbf{q}}(s)\|^2 ds \right). \end{aligned}$$

For some  $t^* \in [0, t]$ , let

$$\|(\xi_u, \xi_{u_t}, \xi_{\mathbf{q}}, \xi_{\sigma})(t^*)\| = \max_{0 \leq s \leq t} \|(\xi_u, \xi_{u_t}, \xi_{\mathbf{q}}, \xi_{\sigma})(s)\|.$$

Then, at  $t = t^*$ , (5.7) becomes

$$\begin{aligned} \|(\xi_u, \xi_{u_t}, \xi_{\mathbf{q}}, \xi_{\sigma})(t^*)\| &\leq \|(\xi_u, \xi_{u_t}, \xi_{\mathbf{q}}, \xi_{\sigma})(0)\| + 2 \int_0^{t^*} \|\eta_{u_{tt}}\| ds \\ &\quad + C(M, \alpha, T) \int_0^{t^*} \|A^{1/2} \xi_{\mathbf{q}}(s)\| ds, \end{aligned}$$

and hence,

$$\begin{aligned} \|(\xi_u, \xi_{u_t}, \xi_{\mathbf{q}}, \xi_{\sigma})(t)\| &\leq \|(\xi_u, \xi_{u_t}, \xi_{\mathbf{q}}, \xi_{\sigma})(t^*)\| \\ &\leq \|(\xi_u, \xi_{u_t}, \xi_{\mathbf{q}}, \xi_{\sigma})(0)\| + 2 \int_0^T \|\eta_{u_{tt}}\| ds \\ &\quad + C(M, \alpha, T) \int_0^T \|A^{1/2} \xi_{\mathbf{q}}(s)\| ds. \end{aligned}$$

Now an application of Gronwall lemma shows that

$$\begin{aligned} (5.8) \quad \|\xi_{u_t}\| + \|A^{1/2} \xi_{\mathbf{q}}\| &\leq C \left( \|\xi_{u_t}(0)\| + \|\xi_{\mathbf{q}}(0)\| + J_1(\xi_{\sigma}(0), \xi_{\sigma}(0))^{1/2} \right. \\ &\quad \left. + J(\xi_u(0), \xi_u(0))^{1/2} + \int_0^T \|\eta_{u_{tt}}\| ds \right). \end{aligned}$$

To estimate  $\|\xi_{\sigma}\|$ , we choose  $\tau_h = \xi_{\sigma}$  in (5.2) and use the Cauchy-Schwarz inequality to arrive at

$$\|\xi_{\sigma}\| \leq C \left( \|A^{1/2} \xi_{\mathbf{q}}\| + \int_0^T \|A^{1/2} \xi_{\mathbf{q}}(s)\| ds \right).$$

A use of (5.8) completes the proof of the lemma.  $\square$

**Proof of Theorem 3.1.** Using the triangle inequality, we can write

$$\|u_t - u_{ht}\| \leq \|u_t - \tilde{u}_{ht}\| + \|\tilde{u}_{ht} - u_{ht}\|.$$

Now a use of Theorem 4.1 and Lemma 5.1 with the choices  $\tilde{u}_h(0) = \Pi_h u_0$ ,  $\tilde{u}_{ht}(0) = \Pi_h u_1$ , and  $\tilde{\mathbf{q}}_h(0) = \mathbf{I}_h \nabla u_0$  yields the estimate (3.22). In the similar way, we can find the estimate (3.23). This completes the rest of the proof.  $\square$

**Remark 5.1.** As a consequence of Lemma 5.1 and the following inequality

$$\|\xi_u(t)\| \leq C \left( \|\xi_u(0)\| + \int_0^t \|\xi_{u_t}\| ds \right),$$

we now obtain an estimate of  $\|\xi_u\|$ . This, in turn, provides the following  $L^\infty(L^2)$  estimate of  $u - u_h$  as

$$\begin{aligned} (5.9) \quad \|u - u_h\|_{L^\infty(L^2(\Omega))} &\leq C \frac{h^{P+D}}{p^{R+S}} \left( \|u_0\|_{H^{r+2}(\mathcal{T}_h)} + \|u_1\|_{H^{r+1}(\mathcal{T}_h)} \right. \\ &\quad \left. + \sum_{j=0}^1 \left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^1(H^{r+2}(\mathcal{T}_h))} + \|u_{tt}\|_{L^1(H^{r+2}(\mathcal{T}_h))} \right). \end{aligned}$$

Note that as a consequence of Theorem 3.1, we obtain estimates (5.9) under the assumption of higher regularity result on the solution. We now use a variant of Baker's nonstandard formulation (see [3]) to provide a proof of  $L^\infty(L^2)$  estimate under reduced regularity result.

Now define the function  $\hat{\phi}$  by

$$\hat{\phi}(t) = \int_0^t \phi(s) ds.$$

After integrating (5.2) and (5.3) with respect to  $t$ , we obtain the new system

$$(5.10) \quad \mathcal{A}(\xi_{\mathbf{q}}, \mathbf{w}_h) - \mathcal{A}_1(\xi_u, \mathbf{w}_h) + J_1(\xi_\sigma, \mathbf{w}_h) = 0 \quad \forall \mathbf{w}_h \in \mathbf{W}_h,$$

$$(5.11) \quad \mathcal{A}_2(\hat{\xi}_{\mathbf{q}}, \tau_h) - \mathcal{A}(\hat{\xi}_\sigma, \tau_h) + \int_0^t \left( \int_0^s \mathcal{B}(s, \tau; \xi_{\mathbf{q}}(\tau), \tau_h) d\tau \right) ds = 0 \quad \forall \tau_h \in \mathbf{W}_h,$$

$$(5.12) \quad (\xi_{u_t}, v_h) + \mathcal{A}_1(v_h, \hat{\xi}_\sigma) + J(\hat{\xi}_u, v_h) = (\eta_{u_t}, v_h) - (e_{ht}(0), v_h) \quad \forall v_h \in V_h.$$

Note that with  $u_{ht}(0) = \Pi_h u_1$ , we have

$$(e_{ht}(0), v_h) = 0, \quad \forall v_h \in V_h.$$

**Proof of Theorem 3.2.** Choose  $\mathbf{w}_h = \hat{\xi}_\sigma$  in (5.10),  $\tau_h = \xi_{\mathbf{q}}$  in (5.11) and  $v_h = \xi_u$  in (5.12). Then adding the resulting equations, we find that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[ \|\xi_u\|^2 + A_2(\hat{\xi}_{\mathbf{q}}, \hat{\xi}_{\mathbf{q}}) + J_1(\hat{\xi}_\sigma, \hat{\xi}_\sigma) + J(\hat{\xi}_u, \hat{\xi}_u) \right] \\ &= (\eta_{u_t}, \xi_u) - \int_0^t \left( \int_0^s \mathcal{B}(s, \tau; \xi_{\mathbf{q}}(\tau), \xi_{\mathbf{q}}) d\tau \right) ds. \end{aligned}$$

Integrating from 0 to  $t$ , and using the non-negativity of  $J$  and  $J_1$ , we arrive at

$$(5.13) \quad \begin{aligned} \|\xi_u\|^2 &+ \|A^{1/2} \hat{\xi}_{\mathbf{q}}\|^2 = \|\xi_u(0)\|^2 + 2 \int_0^t (\eta_{u_t}, \xi_u) ds \\ &- 2 \int_0^t \int_0^s \int_0^\tau \mathcal{B}(\tau, \tau^*; \xi_{\mathbf{q}}(\tau^*), \xi_{\mathbf{q}}(s)) d\tau^* d\tau ds. \end{aligned}$$

Let  $I$  denote the last term on the right hand side of (5.13). Integration by parts yields

$$\begin{aligned} & \int_0^s \int_0^\tau \mathcal{B}(\tau, \tau^*; \xi_{\mathbf{q}}(\tau^*), \xi_{\mathbf{q}}(s)) d\tau^* d\tau = \int_0^s \mathcal{B}(\tau, \tau; \hat{\xi}_{\mathbf{q}}(\tau), \xi_{\mathbf{q}}(s)) d\tau \\ & - \int_0^s \int_0^\tau \mathcal{B}_{\tau^*}(\tau, \tau^*; \hat{\xi}_{\mathbf{q}}(\tau^*), \xi_{\mathbf{q}}(s)) d\tau^* d\tau, \end{aligned}$$

and therefore  $I = -2(I_1 - I_2)$  where

$$I_1 = \int_0^t \int_0^s \mathcal{B}(\tau, \tau; \hat{\xi}_{\mathbf{q}}(\tau), \xi_{\mathbf{q}}(s)) d\tau ds,$$

and

$$I_2 = \int_0^t \int_0^s \int_0^\tau \mathcal{B}_{\tau^*}(\tau, \tau^*; \hat{\xi}_{\mathbf{q}}(\tau^*), \xi_{\mathbf{q}}(s)) d\tau^* d\tau ds.$$

Again, we integrate by parts so that

$$\begin{aligned} I_1 &= \int_0^t \mathcal{B}(s, s; \hat{\xi}_{\mathbf{q}}(s), \hat{\xi}_{\mathbf{q}}(t)) ds - \int_0^t \mathcal{B}(s, s; \hat{\xi}_{\mathbf{q}}(s), \hat{\xi}_{\mathbf{q}}(s)) ds \\ &\leq M \left\{ \|\hat{\xi}_{\mathbf{q}}(t)\| \int_0^t \|\hat{\xi}_{\mathbf{q}}(s)\| ds + \int_0^t \|\hat{\xi}_{\mathbf{q}}(s)\|^2 ds \right\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} I_2 &= \int_0^t \int_0^s \mathcal{B}_\tau(s, \tau; \hat{\xi}_q(\tau), \hat{\xi}_q(t)) d\tau ds - \int_0^t \int_0^s \mathcal{B}_\tau(s, \tau; \hat{\xi}_q(\tau), \hat{\xi}_q(s)) d\tau ds \\ &\leq MT \left\{ \|\hat{\xi}_q(t)\| \int_0^t \|\hat{\xi}_q(s)\| ds + \int_0^t \|\hat{\xi}_q(s)\|^2 ds \right\}. \end{aligned}$$

Using the Cauchy-Schwarz inequality and the bounds for  $I_1$  and  $I_2$ , we obtain

$$\begin{aligned} \|\xi_u(t)\|^2 + \|A^{1/2} \hat{\xi}_q(t)\|^2 &\leq \|\xi_u(0)\|^2 + 2 \int_0^t \|\eta_{u_t}(s)\| \|\xi_u(s)\| ds \\ &\quad + C(M, \alpha, T) \left( \|A^{1/2} \hat{\xi}_q(t)\| \int_0^t \|A^{1/2} \hat{\xi}_q(s)\| ds \right. \\ &\quad \left. + \int_0^t \|A^{1/2} \hat{\xi}_q(s)\|^2 ds \right). \end{aligned}$$

Now, let  $|||(\xi_u, \hat{\xi}_q)(t)|||^2 = \|\xi_u(t)\|^2 + \|A^{1/2} \hat{\xi}_q(t)\|^2$  and

$$|||(\xi_u, \hat{\xi}_q)(t^*)||| = \max_{0 \leq s \leq t} |||(\xi_u, \hat{\xi}_q)(s)|||,$$

for some  $t^* \in [0, t]$ . Then, at  $t = t^*$ , we find that

$$\begin{aligned} |||(\xi_u, \hat{\xi}_q)(t^*)||| &\leq |||(\xi_u, \hat{\xi}_q)(0)||| + 2 \int_0^{t^*} \|\eta_{u_t}(s)\| ds \\ &\quad + C(M, \alpha, T) \int_0^{t^*} |||(\xi_u, \hat{\xi}_q)(s)||| ds, \end{aligned}$$

and therefore,

$$\begin{aligned} |||(\xi_u, \hat{\xi}_q)(t)||| &\leq |||(\xi_u, \hat{\xi}_q)(0)||| + 2 \int_0^t \|\eta_{u_t}(s)\| ds \\ &\quad + C(M, \alpha, T) \int_0^t |||(\xi_u, \hat{\xi}_q)(s)||| ds. \end{aligned}$$

An application of Gronwall lemma yields

$$\|\xi_u(t)\| + \|A^{1/2} \hat{\xi}_q(t)\| \leq C \left( \|\xi_u(0)\| + \int_0^t \|\eta_{u_t}\| ds \right).$$

Finally, a use of the triangle inequality and Theorem 4.1 concludes the proof of Theorem 3.2.  $\square$

## 6. FULLY DISCRETE SCHEME

In this section, we first introduce some notations and formulate the fully discrete DG scheme, then analyze its stability and discuss a priori error estimates.

**6.1. Notations and Scheme.** Let  $k$  ( $0 < k < 1$ ) be the time step,  $k = T/N$  for some positive integer  $N$ , and  $t_n = nk$ . For any function  $\phi$  of time, let  $\phi^n$  denote  $\phi(t_n)$ . We shall use this notation for functions defined for continuous in time as well as those defined for discrete in time. We let

$$U^{n+1/2} = \frac{U^{n+1} + U^n}{2}, \quad U^{n;1/4} = \frac{U^{n+1} + 2U^n + U^{n-1}}{4} = \frac{U^{n+1/2} + U^{n-1/2}}{2},$$

and define the following terms for the discrete temporal derivatives:

$$\begin{aligned}\partial_t U^{n+1/2} &= \frac{U^{n+1} - U^n}{k}, & \bar{\partial}_t U^{n+1/2} &= \frac{U^{n+1/2} - U^{n-1/2}}{k}, \\ \delta_t U^n &= \frac{U^{n+1} - U^{n-1}}{2k} = \frac{\partial_t U^{n+1/2} + \partial_t U^{n-1/2}}{2},\end{aligned}$$

and

$$\partial_t^2 U^n = \frac{U^{n+1} - 2U^n + U^{n-1}}{2k} = \frac{\partial_t U^{n+1/2} - \partial_t U^{n-1/2}}{k}.$$

The discrete-in-time scheme is based on a symmetric difference approximation around the nodal points, and integral terms are computed by using the second order quadrature formula

$$\epsilon^n(\phi) = k \sum_{j=0}^{n-1} \phi(t_{j+1/2}) \approx \int_0^{t_n} \phi(s) ds, \quad \text{with } t_{j+1/2} = (j + 1/2)k.$$

Thus, the discrete-in-time scheme for the problem (3.1)-(3.3) is to seek  $(U^n, \mathbf{Q}^n, \mathbf{Z}^n) \in V_h \times \mathbf{W}_h \times \mathbf{W}_h$ , such that

$$(6.1) \quad \frac{2}{k}(\partial_t U^{1/2}, v_h) + \mathcal{A}_1(v_h, \mathbf{Z}^{1/2}) + J(U^{1/2}, v_h) = (f^{1/2} + \frac{2}{k}u_1, v_h),$$

$$(6.2) \quad \mathcal{A}(\mathbf{Q}^{n+1/2}, \mathbf{w}_h) - \mathcal{A}_1(U^{n+1/2}, \mathbf{w}_h) + J_1(\mathbf{Z}^{n+1/2}, \mathbf{w}_h) = 0, \quad n \geq 0,$$

$$(6.3) \quad \mathcal{A}_2(\mathbf{Q}^{n+1/2}, \boldsymbol{\tau}_h) - \mathcal{A}(\mathbf{Z}^{n+1/2}, \boldsymbol{\tau}_h) + \epsilon^{n+1/2}(\mathcal{B}^{n+1/2}(\mathbf{Q}, \boldsymbol{\tau}_h)) = 0, \quad n \geq 0,$$

$$(6.4) \quad (\partial_t^2 U^n, v_h) + \mathcal{A}_1(v_h, \mathbf{Z}^{n;1/4}) + J(U^{n;1/4}, v_h) = (f^{n;1/4}, v_h), \quad n \geq 1,$$

for all  $(v_h, \boldsymbol{\tau}_h, \mathbf{w}_h) \in V_h \times \mathbf{W}_h \times \mathbf{W}_h$ , with given  $(U^0, \mathbf{Q}^0, \mathbf{Z}^0) \in V_h \times \mathbf{W}_h \times \mathbf{W}_h$ . In (6.3),

$$\epsilon^{n+1/2}(\mathcal{B}^{n+1/2}(\mathbf{Q}, \boldsymbol{\tau}_h)) = \frac{1}{2} [\epsilon^{n+1}(\mathcal{B}^{n+1}(\mathbf{Q}, \boldsymbol{\tau}_h)) + \epsilon^n(\mathcal{B}^n(\mathbf{Q}, \boldsymbol{\tau}_h))],$$

where

$$\epsilon^n(\mathcal{B}^n(\mathbf{Q}, \chi)) = k \sum_{j=0}^{n-1} \mathcal{B}(t_n, t_{j+1/2}; \mathbf{Q}^{j+1/2}, \chi).$$

This choice of the time discretization leads to a second order accuracy in  $k$ .

**6.2. Stability of the Discrete Problem.** We let  $\Phi^n = (U^n, \mathbf{Q}^n, \mathbf{Z}^n)$  and define the discrete energy norm

$$|||\Phi^{n+1/2}|||^2 = ||\partial_t U^{n+1/2}||^2 + ||A^{1/2} \mathbf{Q}^{n+1/2}||^2 + J_1(\mathbf{Z}^{n+1/2}, \mathbf{Z}^{n+1/2}) + J(U^{n+1/2}, U^{n+1/2}).$$

For the purpose of later error analysis, we shall first derive a stability result for a modified scheme, namely;

$$(6.5) \quad \mathcal{A}(\mathbf{Q}^{n+1/2}, \mathbf{w}_h) - \mathcal{A}_1(U^{n+1/2}, \mathbf{w}_h) + J_1(\mathbf{Z}^{n+1/2}, \mathbf{w}_h) = 0,$$

$$(6.6) \quad \mathcal{A}_2(\mathbf{Q}^{n+1/2}, \boldsymbol{\tau}_h) - \mathcal{A}(\mathbf{Z}^{n+1/2}, \boldsymbol{\tau}_h) + \epsilon^{n+1/2}(\mathcal{B}^{n+1/2}(\mathbf{Q}, \boldsymbol{\tau}_h)) = G^{n+1/2}(\boldsymbol{\tau}_h),$$

$$(6.7) \quad (\partial_t^2 U^n, v_h) + \mathcal{A}_1(v_h, \mathbf{Z}^{n;1/4}) + J(U^{n;1/4}, v_h) = (f^{n;1/4}, v_h) + F^n(v_h),$$

where  $F^n$  and  $G^n$  are linear functionals on  $V_h$  and  $\mathbf{W}_h$ , respectively, with  $G^0 = 0$ . Set

$$||F^n|| = \sup_{\chi \in V_h, \chi \neq 0} \frac{|F^n(\chi)|}{||\chi||},$$

and similarly define the norm for  $G^n$ . Then, the following stability result holds.

**Theorem 6.1.** *There exist positive constants  $C$  and  $k_0$  such that for  $0 < k \leq k_0$ ,  $m \geq 1$ ,  $t_{m+1} \leq T$ , the solution of the fully discrete problem (6.5)-(6.7) satisfies the following stability estimate*

$$(6.8) \quad |||\Phi^{m+1/2}||| \leq C \left\{ |||\Phi^{1/2}||| + k \sum_{n=0}^{m+1} \|f^n\| + k \sum_{n=1}^m \|F^n\| + k \sum_{n=0}^m \|\partial_t G^{n+1/2}\| \right\}.$$

*Proof.* We first subtract (6.5) from itself with  $n + 1/2$  replaced by  $n - 1/2$  and take the average of (6.6) at two different times. Then, we choose  $\mathbf{w}_h = \mathbf{Z}^{n;1/4}$  in (6.5),  $\boldsymbol{\tau}_h = \delta_t \mathbf{Q}^n$  in (6.6), and  $v_h = \delta_t U^n$  in (6.7) to obtain

$$(6.9) \quad \mathcal{A}(\delta_t \mathbf{Q}^n, \mathbf{Z}^{n;1/4}) - \mathcal{A}_1(\delta_t U^n, \mathbf{Z}^{n;1/4}) + J_1(\delta_t \mathbf{Z}^n, \mathbf{Z}^{n;1/4}) = 0,$$

$$(6.10) \quad \begin{aligned} \mathcal{A}_2(\mathbf{Q}^{n;1/4}, \delta_t \mathbf{Q}^n) - \mathcal{A}(\mathbf{Z}^{n;1/4}, \delta_t \mathbf{Q}^n) + \frac{1}{2} \left( \epsilon^{n+1/2} (\mathcal{B}^{n+1/2}(\mathbf{Q}, \delta_t \mathbf{Q}^n)) \right. \\ \left. + \epsilon^{n-1/2} (\mathcal{B}^{n-1/2}(\mathbf{Q}, \delta_t \mathbf{Q}^n)) \right) = G^{n;1/4}(\delta_t \mathbf{Q}^n), \end{aligned}$$

$$(6.11) \quad \begin{aligned} (\partial_t^2 U^n, \delta_t U^n) + \mathcal{A}_1(\delta_t U^n, \mathbf{Z}^{n;1/4}) + J(U^{n;1/4}, \delta_t U^n) \\ = (f^{n;1/4}, \delta_t U^n) + F^n(\delta_t U^n). \end{aligned}$$

By adding, we find that

$$(6.12) \quad \begin{aligned} (\partial_t^2 U^n, \delta_t U^n) + \mathcal{A}_2(\mathbf{Q}^{n;1/4}, \delta_t \mathbf{Q}^n) + J_1(\delta_t \mathbf{Z}^n, \mathbf{Z}^{n;1/4}) + J(U^{n;1/4}, \delta_t U^n) \\ = (f^{n;1/4}, \delta_t U^n) + F^n(\delta_t U^n) - \frac{1}{2} \left( \epsilon^{n+1/2} (\mathcal{B}^{n+1/2}(\mathbf{Q}, \delta_t \mathbf{Q}^n)) \right. \\ \left. + \epsilon^{n-1/2} (\mathcal{B}^{n-1/2}(\mathbf{Q}, \delta_t \mathbf{Q}^n)) \right) + G^{n;1/4}(\delta_t \mathbf{Q}^n) \\ = I_1^n + I_2^n + I_3^n + I_4^n. \end{aligned}$$

Notice that

$$(\partial_t^2 U^n, \delta_t U^n) = \frac{1}{2k} \left( \|\partial_t U^{n+1/2}\|^2 - \|\partial_t U^{n-1/2}\|^2 \right),$$

and

$$\mathcal{A}_2(\mathbf{Q}^{n;1/4}, \delta_t \mathbf{Q}^n) = \frac{1}{2k} \left( \mathcal{A}_2(\mathbf{Q}^{n+1/2}, \mathbf{Q}^{n+1/2}) - \mathcal{A}_2(\mathbf{Q}^{n-1/2}, \mathbf{Q}^{n-1/2}) \right).$$

Now, we multiply both sides of (6.12) by  $2k$  and sum from  $n = 2$  to  $m$ , to obtain

$$(6.13) \quad |||\Phi^{m+1/2}|||^2 \leq |||\Phi^{3/2}|||^2 + 2k \left| \sum_{n=2}^m (I_1^n + I_2^n + I_3^n + I_4^n) \right|.$$

Define for some  $m^*$  with  $0 \leq m^* \leq m$ ,

$$|||\Phi^{m^*+1/2}||| = \max_{0 \leq n \leq m} |||\Phi^{n+1/2}|||.$$

An application of the Cauchy-Schwarz inequality to (6.13) yields

$$\begin{aligned} k \left| \sum_{n=2}^m (I_1^n + I_2^n) \right| &\leq \frac{k}{2} \sum_{n=2}^m \left( \|f^{n;1/4}\| + \|F^n\| \right) \left( \|\partial_t U^{n+1/2}\| + \|\partial_t U^{n-1/2}\| \right) \\ &\leq k \sum_{n=2}^m \left( \|f^{n;1/4}\| + \|F^n\| \right) |||\Phi^{m^*+1/2}|||. \end{aligned}$$

Next, we set

$$\tilde{B}_{j+1/2}^{n+1/2} = \frac{1}{2} (B(t_{n+1}, t_{j+1/2}) + B(t_n, t_{j+1/2})).$$

In order to estimate  $I_3^n$ , we write

$$\begin{aligned}
& \epsilon^{n+1/2}(\mathcal{B}^{n+1/2}(\mathbf{Q}, \delta_t \mathbf{Q}^n)) \\
&= \frac{k}{2} \left[ \sum_{j=0}^n \mathcal{B}(t_{n+1}, t_{j+1/2}; \mathbf{Q}^{j+1/2}, \delta_t \mathbf{Q}^n) + \sum_{j=0}^{n-1} \mathcal{B}(t_n, t_{j+1/2}; \mathbf{Q}^{j+1/2}, \delta_t \mathbf{Q}^n) \right] \\
&= \frac{1}{2} \int_{\Omega} B(t_{n+1}, t_{n+1/2}) \mathbf{Q}^{n+1/2} \cdot (\mathbf{Q}^{n+1/2} - \mathbf{Q}^{n-1/2}) dx \\
&\quad + \sum_{j=0}^{n-1} \int_{\Omega} \tilde{B}_{j+1/2}^{n+1/2} \mathbf{Q}^{j+1/2} \cdot (\mathbf{Q}^{n+1/2} - \mathbf{Q}^{n-1/2}) dx \\
&= T_1^n + T_2^n.
\end{aligned}$$

The first term  $T_1$  can be bounded as follows

$$|T_1^n| \leq \frac{M}{2} \left( \|\mathbf{Q}^{n+1/2}\|^2 + \|\mathbf{Q}^{n+1/2}\| \|\mathbf{Q}^{n-1/2}\| \right)$$

showing that

$$(6.14) \quad k \left| \sum_{n=2}^m T_1^n \right| \leq \frac{Mk}{2\alpha} \|A^{1/2} \mathbf{Q}^{m+1/2}\|^2 + C(M, \alpha) k \left( \sum_{n=1}^{m-1} \|A^{1/2} \mathbf{Q}^{n+1/2}\| \right) |||\Phi^{m^*+1/2}|||.$$

For the second term  $T_2^n$ , we use the fact that

$$(6.15) \quad H^{n+1/2} \bar{\partial}_t \mathbf{Q}^{n+1/2} = \bar{\partial}_t (H^{n+1/2} \mathbf{Q}^{n+1/2}) - \bar{\partial}_t (H^{n+1/2}) \mathbf{Q}^{n-1/2},$$

and obtain after summation

$$\begin{aligned}
T_2^n &= k \sum_{j=0}^{n-1} \int_{\Omega} \bar{\partial}_t (\tilde{B}_{j+1/2}^{n+1/2} \mathbf{Q}^{n+1/2}) \cdot \mathbf{Q}^{j+1/2} dx - k \sum_{j=0}^{n-1} \int_{\Omega} \bar{\partial}_t (\tilde{B}_{j+1/2}^{n+1/2}) \mathbf{Q}^{j+1/2} \cdot \mathbf{Q}^{n-1/2} dx \\
&= k \bar{\partial}_t \left( \sum_{j=0}^{n-1} \int_{\Omega} \tilde{B}_{j+1/2}^{n+1/2} \mathbf{Q}^{n+1/2} \cdot \mathbf{Q}^{j+1/2} dx \right) + \int_{\Omega} \tilde{B}_{n-1/2}^{n-1/2} \mathbf{Q}^{n-1/2} \mathbf{Q}^{n-1/2} dx \\
&\quad - k \sum_{j=0}^{n-1} \int_{\Omega} \bar{\partial}_t (\tilde{B}_{j+1/2}^{n+1/2}) \mathbf{Q}^{j+1/2} \cdot \mathbf{Q}^{n-1/2} dx.
\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}
k \left| \sum_{n=2}^m T_2^n \right| &\leq k^2 \left| \sum_{j=0}^{m-1} \int_{\Omega} \tilde{B}_{j+1/2}^{m+1/2} \mathbf{Q}^{j+1/2} \cdot \mathbf{Q}^{m+1/2} dx - \int_{\Omega} \tilde{B}_{1/2}^{3/2} \mathbf{Q}^{3/2} \cdot \mathbf{Q}^{1/2} dx \right| \\
&\quad + k \left| \sum_{n=2}^m \int_{\Omega} \tilde{B}_{n-1/2}^{n-1/2} \mathbf{Q}^{n-1/2} \cdot \mathbf{Q}^{n-1/2} dx \right| \\
&\quad + k^2 \left| \sum_{n=2}^m \sum_{j=0}^{n-1} \int_{\Omega} \bar{\partial}_t \left( \tilde{B}_{j+1/2}^{n+1/2} \right) \mathbf{Q}^{j+1/2} \cdot \mathbf{Q}^{n-1/2} dx \right| \\
&\leq Mk^2 \|\mathbf{Q}^{m+1/2}\| \sum_{j=0}^{m-1} \|\mathbf{Q}^{j+1/2}\| + Mk^2 \|\mathbf{Q}^{3/2}\| \|\mathbf{Q}^{1/2}\| \\
&\quad + Mk \sum_{n=2}^m \|\mathbf{Q}^{n-1/2}\|^2 + MTk \sum_{j=0}^{m-1} \|\mathbf{Q}^{j+1/2}\|^2 \\
&\leq C(M, \alpha, T)k \left( \sum_{n=0}^{m-1} \|A^{1/2} \mathbf{Q}^{n+1/2}\| \right) |||\Phi^{m^*+1/2}|||.
\end{aligned}$$

We can now estimate  $\epsilon^{n-1/2}(\mathcal{B}^{n-1/2}(\mathbf{Q}, \delta_t \mathbf{Q}^n))$  in a similar way, but without having the term  $\|A^{1/2} \mathbf{Q}^{m+1/2}\|^2$  on the right hand side of (6.14), and thus, obtain

$$k \left| \sum_{n=2}^m I_3^n \right| \leq \frac{Mk}{4\alpha} \|A^{1/2} \mathbf{Q}^{m+1/2}\|^2 + C(M, \alpha, T)k \left( \sum_{n=0}^{m-1} \|A^{1/2} \mathbf{Q}^{n+1/2}\| \right) |||\Phi^{m^*+1/2}|||.$$

For the last term  $I_4 = kG^{n;1/4}(\delta_t \mathbf{Q}^n)$ , we again use summation by parts technique to arrive at

$$k \left| \sum_{n=2}^m I_4^n \right| \leq \left| G^{m;1/4}(\mathbf{Q}^{m+1/2}) - G^{1;1/4}(\mathbf{Q}^{3/2}) - k \sum_{n=2}^m \bar{\partial}_t(G^{n;1/4})(\mathbf{Q}^{n-1/2}) \right|.$$

Notice that, since  $G^0 = 0$ , it follows that

$$G^m = k \sum_{n=0}^{m-1} \partial_t G^{n+1/2},$$

and hence, we obtain

$$k \left| \sum_{n=2}^m I_4^n \right| \leq C(\alpha)k \left( \sum_{n=0}^m \|\partial_t G^{n+1/2}\| \right) |||\Phi^{m^*+1/2}|||.$$

It remains now to bound the term  $|||\Phi^{3/2}|||$  on the right hand side of (6.13). Equation (6.12) taken at  $n = 1$  yields

$$\begin{aligned}
|||\Phi^{3/2}|||^2 &\leq |||\Phi^{1/2}|||^2 + 2k \left| (f^{1;1/4}, \delta_t U^1) + F^1(\delta_t U^1) - \frac{1}{2} \left( \epsilon^{3/2}(\mathcal{B}^{3/2}(\mathbf{Q}, \delta_t \mathbf{Q}^1)) \right. \right. \\
&\quad \left. \left. + \epsilon^{1/2}(\mathcal{B}^{1/2}(\mathbf{Q}, \delta_t \mathbf{Q}^1)) + G^{1;1/4}(\delta_t \mathbf{Q}^1) \right) \right| \\
(6.16) \quad &\leq |||\Phi^{1/2}|||^2 + C(M, \alpha)k \left( \|f^{1;1/4}\| + \|F^1\| + \|A^{1/2} \mathbf{Q}^{1/2}\| \right. \\
&\quad \left. + \|A^{1/2} \mathbf{Q}^{3/2}\| + \|\partial_t G^{1/2}\| + \|\partial_t G^{3/2}\| \right) |||\Phi^{m^*+1/2}|||.
\end{aligned}$$



Now, substitute estimates involving  $I_1^n, \dots, I_4^n$  and (6.16) in (6.13) to find that

$$(6.17) \quad \left(1 - \frac{M}{2\alpha}k\right) |||\Phi^{m+1/2}|||^2 \leq |||\Phi^{1/2}|||^2 + Ck \left\{ \sum_{n=0}^{m+1} \|f^n\| + \sum_{n=1}^m \|F^n\| \right. \\ \left. + \sum_{n=0}^m \|\partial_t G^{n+1/2}\| + \sum_{n=0}^{m-1} |||\Phi^{n+1/2}||| \right\} |||\Phi^{m^*+1/2}|||,$$

Choose  $k_0 > 0$  such that for  $0 < k \leq k_0$ ,  $(1 - \frac{M}{2\alpha}k) > 0$ . Then replace  $m$  by  $m^*$  in (6.17) to obtain

$$(6.18) \quad |||\Phi^{m^*+1/2}||| \leq C \left\{ |||\Phi^{1/2}||| + k \sum_{n=0}^{m^*+1} \|f^n\| + k \sum_{n=1}^{m^*} \|F^n\| \right. \\ \left. + k \sum_{n=0}^{m^*} \|\partial_t G^{n+1/2}\| + k \sum_{n=0}^{m^*-1} |||\Phi^{n+1/2}||| \right\},$$

and hence, replacing  $m^*$  by  $m$  on the right hand side of (6.18), it follows that

$$(6.19) \quad |||\Phi^{m+1/2}||| \leq |||\Phi^{m^*+1/2}||| \leq C \left\{ |||\Phi^{1/2}||| + k \sum_{n=0}^{m+1} \|f^n\| + k \sum_{n=1}^m \|F^n\| \right. \\ \left. + k \sum_{n=0}^m \|\partial_t G^{n+1/2}\| + k \sum_{n=0}^{m-1} |||\Phi^{n+1/2}||| \right\}.$$

An application of the discrete Gronwall lemma to (6.19) completes the rest of the proof.  $\square$

**6.3. Convergence Analysis.** For  $\phi \in \mathbf{W}_h$ , we define a linear functional  $\mathcal{E}_{\mathcal{B}}^n(\phi)$  representing the error in the quadrature formula by

$$\mathcal{E}_{\mathcal{B}}^n(\phi)(\chi) = \epsilon^n(\mathcal{B}^n(\phi, \chi)) - \int_0^{t_n} \mathcal{B}(t_n, s; \phi, \chi) ds.$$

Notice that  $\mathcal{E}_{\mathcal{B}}^0(\phi) = 0$ . In our analysis, we shall use the following lemma which can be found in [16].

**Lemma 6.2.** *There exists a positive constant  $C$ , independent of  $h$  and  $k$  such that the following estimates*

$$k \sum_{n=0}^m \|\mathcal{E}_{\mathcal{B}}^{n+1}(\tilde{\mathbf{q}}_h)\| \leq Ck^2 \int_0^{t_{m+1}} (||\tilde{\mathbf{q}}_h|| + ||\tilde{\mathbf{q}}_{ht}|| + ||\tilde{\mathbf{q}}_{htt}||) ds,$$

and

$$k \sum_{n=0}^m \|\partial_t \mathcal{E}_{\mathcal{B}}^{n+1/2}(\tilde{\mathbf{q}}_h)\| \leq Ck^2 \int_0^{t_{m+1}} (||\tilde{\mathbf{q}}_h|| + ||\tilde{\mathbf{q}}_{ht}|| + ||\tilde{\mathbf{q}}_{htt}||) ds,$$

hold.

In order to derive *a priori* error estimates for the fully discrete scheme, we rewrite

$$\begin{aligned} u^n - U^n &:= (u^n - \tilde{u}_h^n) - (U^n - \tilde{u}_h^n) =: \eta_U^n - \xi_U^n, \\ \mathbf{q}^n - \mathbf{Q}^n &:= (\mathbf{q}^n - \tilde{\mathbf{q}}_h^n) - (\mathbf{Q}^n - \tilde{\mathbf{q}}_h^n) =: \eta_Q^n - \xi_Q^n, \\ \boldsymbol{\sigma}^n - \mathbf{Z}^n &:= (\boldsymbol{\sigma}^n - \tilde{\boldsymbol{\sigma}}_h^n) - (\mathbf{Z}^n - \tilde{\boldsymbol{\sigma}}_h^n) =: \eta_Z^n - \xi_Z^n. \end{aligned}$$

The following theorem provides *a priori* error estimates for the fully discrete scheme.

**Theorem 6.3.** *Let  $(u, \mathbf{q}, \boldsymbol{\sigma})$  be the solution of (3.1)-(3.3). Further, let  $(U^n, \mathbf{Q}^n, \mathbf{Z}^n) \in V_h \times \mathbf{W}_h \times \mathbf{W}_h$  be the solution of (6.1)-(6.4). Assume that  $U^0 = \tilde{u}_0$ ,  $\mathbf{Q}^0 = \mathbf{I}_h \nabla u_0$  and  $\mathbf{Z}^0 = \mathbf{I}_h (A \nabla u_0)$ . Then there exists constants  $C > 0$ , independent of  $h$  and  $k$ , and  $k_0 > 0$ , such that for  $0 < k < k_0$  and  $m = 0, 1, \dots, N-1$*

$$(6.20) \quad \|\partial_t(u(t_{m+1/2}) - U^{m+1/2})\|_{L^2(\Omega)} \leq C \frac{h^{P+D}}{p^{R+S}} \left( \|u_0\|_{H^{r+2}(\mathcal{T}_h)} + \|u_1\|_{H^{r+2}(\mathcal{T}_h)} \right. \\ \left. + \sum_{j=0}^2 \left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^1(H^{r+2}(\mathcal{T}_h))} \right) + Ck^2 \sum_{j=3}^4 \int_0^{t_{m+1}} \left\| \frac{\partial^j u}{\partial t^j}(s) \right\| ds,$$

and

$$(6.21) \quad \|\mathbf{q}(t_{m+1/2}) - \mathbf{Q}^{m+1/2}\|_{L^2(\Omega)} + \|\boldsymbol{\sigma}(t_{m+1/2}) - \mathbf{Z}^{m+1/2}\|_{L^2(\Omega)} \leq \\ C \frac{h^P}{p^R} \left( \|u_0\|_{H^{r+2}(\Omega_h)} + \|u_1\|_{H^{r+2}(\Omega_h)} + \sum_{j=0}^2 \left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^1(H^{r+2}(\Omega_h))} \right) \\ + Ck^2 \sum_{j=3}^4 \int_0^{t_{m+1}} \left\| \frac{\partial^j u}{\partial t^j}(s) \right\| ds,$$

where  $P = \min\{r + \frac{1}{2}(1 + \mu_*), p + \frac{1}{2}(1 - \mu^*)\}$ ,  $D = \frac{1}{2}(1 + \mu_*)$ ,  $R = r + \min\{\mu_*, 1 - \mu^*\}$  and  $S = \min\{\frac{1}{2}, \mu_*\}$ .

*Proof.* Since estimates for  $\eta_U$ ,  $\boldsymbol{\eta}_Q$  and  $\boldsymbol{\eta}_Z$  are known from Theorem 4.1, it is enough to estimate  $\xi_U$ ,  $\boldsymbol{\xi}_Q$  and  $\boldsymbol{\xi}_Z$ . Using (4.1)-(4.3), we derive the following system

$$(6.22) \quad \frac{2}{k}(\partial_t \xi_U^{1/2}, v_h) + \mathcal{A}_1(v_h, \boldsymbol{\xi}_Z^{1/2}) + J(\xi_U^{1/2}, v_h) = \frac{2}{k}(\partial_t \eta_U^{1/2}, v_h) + (2r^0, v_h)$$

$$(6.23) \quad \mathcal{A}(\boldsymbol{\xi}_Q^{n+1/2}, \mathbf{w}_h) - \mathcal{A}_1(\xi_U^{n+1/2}, \mathbf{w}_h) + J_1(\boldsymbol{\xi}_Z^{n+1/2}, \mathbf{w}_h) = 0,$$

$$(6.24) \quad \mathcal{A}_2(\boldsymbol{\xi}_Q^{n+1/2}, \boldsymbol{\tau}_h) - \mathcal{A}(\boldsymbol{\xi}_Z^{n+1/2}, \boldsymbol{\tau}_h) + \epsilon^{n+1/2}(\mathcal{B}^{n+1/2}(\boldsymbol{\xi}_Q, \boldsymbol{\tau}_h)) \\ = -\mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)(\boldsymbol{\tau}_h)$$

$$(6.25) \quad (\partial_t^2 \xi_U^n, v_h) + \mathcal{A}_1(v_h, \boldsymbol{\xi}_Z^{n;1/4}) + J(\xi_U^{n;1/4}, v_h) = (\partial_t^2 \eta_U^n, v_h) + (r^n, v_h),$$

where  $r^0 = \frac{1}{2}u_{tt}^{1/2} + \frac{1}{k}(u_1 - \partial_t u^{1/2})$ , and

$$r^n = u_{tt}^{n;1/4} - \partial_t^2 u^n = \frac{1}{12} \int_{-k}^k (|t| - k) (3 - 2(1 - |t|/k)^2) \frac{\partial^4 u}{\partial t^4}(t^n + t) dt, \quad n \geq 1.$$

With  $F^n(v_h) = (\partial_t^2 \eta_U^n, v_h) + (r^n, v_h)$  and  $G^n(\boldsymbol{\tau}_h) = \mathcal{E}_B^n(\tilde{\mathbf{q}}_h)(\boldsymbol{\tau}_h)$ , we apply Theorem 6.1 to arrive at

$$\|\Psi^{m+1/2}\| \leq C \left\{ \|\Psi^{1/2}\| + k \sum_{n=1}^m (\|\partial_t^2 \eta_U^n\| + \|r^n\|) + k \sum_{n=0}^m \|\partial_t \mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)\| \right\},$$

where  $\Psi^n = (\xi_U^n, \boldsymbol{\xi}_Q^n, \boldsymbol{\xi}_Z^n)$ . To estimate  $\Psi^{1/2}$  on the right hand side of this inequality, we first note that,  $\xi_U^0 = 0$  since  $U^0 = \tilde{u}_0$ , and hence,  $\xi_U^{1/2} = \frac{k}{2} \partial_t \xi_U^{1/2}$ . Now, choose  $v_h = \xi_U^{1/2}$  in (6.22),  $v_h = \boldsymbol{\xi}_Z^{1/2}$  in (6.23), and  $v_h = \boldsymbol{\xi}_Q^{1/2}$  in (6.24). Adding the resulting equations, and

taking into account that  $\epsilon^0(\phi) = 0$  and  $\mathcal{E}_B^0(\phi) = 0$ , we obtain

$$\begin{aligned} |||\Psi^{1/2}|||^2 &\leq \left| \left( \partial_t \eta_U^{1/2}, \partial_t \xi_U^{1/2} \right) \right| + k \left| \left( r^0, \partial_t \xi_U^{1/2} \right) \right| + \frac{1}{2} \left| \epsilon^1(\mathcal{B}^1(\xi_Q, \xi_Q^{1/2})) \right| + \frac{1}{2} \left| \mathcal{E}_B^1(\tilde{\mathbf{q}}_h)(\xi_Q^{1/2}) \right| \\ &\leq C(\alpha) \left( \|\partial_t \eta_U^{1/2}\| + k\|r^0\| + \|\mathcal{E}_B^1(\tilde{\mathbf{q}}_h)\| \right) |||\Psi^{1/2}||| + \frac{Mk}{2\alpha} \|A^{1/2} \xi_Q^{1/2}\|^2 \\ &\leq C(\alpha) \left( \|\partial_t \eta_U^{1/2}\| + k\|r^0\| + \|\mathcal{E}_B^1(\tilde{\mathbf{q}}_h)\| \right) |||\Psi^{1/2}||| + \frac{Mk}{2\alpha} |||\Psi^{1/2}|||^2. \end{aligned}$$

For  $0 < k \leq k_0$ ,  $(1 - (Mk)/(2\alpha)) > 0$  and hence,

$$|||\Psi^{1/2}||| \leq C \left\{ \|\partial_t \eta_U^{1/2}\| + k\|r^0\| + \|\mathcal{E}_B^1(\tilde{\mathbf{q}}_h)\| \right\},$$

which shows that

$$(6.26) \quad |||\Psi^{m+1/2}||| \leq C \left\{ \|\partial_t \eta_U^{1/2}\| + k \sum_{n=1}^m \|\partial_t^2 \eta_U^n\| + k \sum_{n=0}^m \|r^n\| + k \sum_{n=0}^m \|\partial_t \mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)\| \right\}.$$

To estimate the terms on the right hand side of (6.26), we note that

$$(6.27) \quad \|\partial_t \eta_U^{1/2}\| \leq \frac{1}{k} \int_0^k \|\eta_{Ut}(s)\| ds,$$

and by Taylor series expansions, we arrive at

$$\begin{aligned} k \sum_{n=1}^m \|\partial_t^2 \eta_U^n\| &\leq \frac{1}{k} \sum_{n=1}^m \left\{ \int_{t_n}^{t_{n+1}} (t_{n+1} - s) \|\eta_{Utt}(s)\| ds + \int_{t_{n-1}}^{t_n} (s - t_{n-1}) \|\eta_{Utt}(s)\| ds \right\} \\ (6.28) \quad &\leq 2 \int_0^{t_{m+1}} \|\eta_{Utt}(s)\| ds. \end{aligned}$$

Further, we find that

$$\|r^n\| \leq Ck \int_{t_{n-1}}^{t_{n+1}} \left\| \frac{\partial^4 u}{\partial t^4}(s) \right\| ds, \quad n \geq 1,$$

and

$$\|r^0\| \leq Ck \|u_{ttt}\|_{L^\infty(0, k/2; L^2(\Omega))} \leq Ck \int_0^{t_{m+1}} \left( \left\| \frac{\partial^3 u}{\partial t^3}(s) \right\| + \left\| \frac{\partial^4 u}{\partial t^4}(s) \right\| \right) ds.$$

Hence,

$$(6.29) \quad k \sum_{n=0}^m \|r^n\| \leq Ck^2 \int_0^{t_{m+1}} \left( \left\| \frac{\partial^3 u}{\partial t^3}(s) \right\| + \left\| \frac{\partial^4 u}{\partial t^4}(s) \right\| \right) ds.$$

For the last term in (6.26), a use of Lemma 6.2 with the triangle inequality yields

$$k \sum_{n=0}^m \|\partial_t \mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)\| \leq Ck^2 \sum_{j=0}^2 \int_0^{t_{m+1}} \left( \left\| \frac{\partial^j \mathbf{q}}{\partial t^j}(s) \right\| + \left\| \frac{\partial^j \eta_{\mathbf{q}}}{\partial t^j}(s) \right\| \right) ds,$$

and hence, using the estimates in Theorem 4.1, we deduce that

$$(6.30) \quad k \sum_{n=0}^m \|\partial_t \mathcal{E}_B^{n+1/2}(\tilde{\mathbf{q}}_h)\| \leq Ck^2 \sum_{j=0}^2 \int_0^{t_{m+1}} \left\| \frac{\partial^j u}{\partial t^j}(s) \right\|_{H^{r+2}(\Omega)} ds.$$

Substitute (6.27)-(6.30) in (6.26) and use the triangle inequality with Theorem 4.1 we obtain the error estimates involving  $u$  and  $\mathbf{q}$  in (6.20)-(6.21). Now to complete the proof of (6.21),

it remains to estimate  $\|\xi_Z^{m+1/2}\|$ . To do so, choose  $\tau_h = \xi_Z^{m+1/2}$  in (6.24), and conclude that

$$\|\xi_Z^{m+1/2}\| \leq C(M, T) \left( \max_{0 \leq n \leq m} \|\xi_Q^{n+1/2}\| + \|\mathcal{E}_B^{m+1/2}(\tilde{q}_h)\| \right).$$

Finally, a use of the triangle inequality and Theorem 4.1 completes the rest of the proof.  $\square$

Below, we again recall a variant of Baker's nonstandard energy formulation to prove  $L^2$ -estimate for the error under reduced regularity conditions. We shall introduce the following notations for proving the final theorem of this paper. Define

$$\hat{\phi}^0 = 0, \quad \hat{\phi}^n = k \sum_{j=0}^{n-1} \phi^{j+1/2}.$$

Then,

$$\partial_t \hat{\phi}^{n+1/2} = \phi^{n+1/2},$$

and

$$k \sum_{j=1}^n \phi^{j;1/4} = \hat{\phi}^{n+1/2} - \frac{k}{2} \phi^{1/2}.$$

Let  $R^n = k \sum_{j=0}^n r^n$ . Multiplying (6.24) and (6.25) by  $k$  and summing over  $n$ , we derive the new system

$$(6.31) \quad \mathcal{A}(\xi_Q^{n+1/2}, \mathbf{w}_h) - \mathcal{A}_1(\xi_U^{n+1/2}, \mathbf{w}_h) + J_1(\xi_Z^{n+1/2}, \mathbf{w}_h) = 0,$$

$$(6.32) \quad \mathcal{A}_2(\hat{\xi}_Q^{n+1/2}, \tau_h) - \mathcal{A}(\hat{\xi}_Z^{n+1/2}, \tau_h) + \hat{\epsilon}^{n+1/2}(\mathcal{B}(\xi_Q, \tau_h)) \\ = -\hat{\mathcal{E}}_B^{n+1/2}(\tilde{q}_h)(\tau_h)$$

$$(6.33) \quad (\partial_t \xi_U^{n+1/2}, v_h) + \mathcal{A}_1(v_h, \hat{\xi}_Z^{n+1/2}) + J(\xi_U^{n+1/2}, v_h) = (\partial_t \eta_U^{n+1/2}, v_h) + (R^n, v_h),$$

where the meanings of  $\hat{\epsilon}^{n+1/2}(\mathcal{B}(\xi_Q, \tau_h))$  and  $\hat{\mathcal{E}}_B^{n+1/2}(\tilde{q}_h)(\tau_h)$  are obvious. Notice that (6.33) is derived after cancellation using (6.22). Also, remark that (6.33) reduces to (6.22) when  $n = 0$ . Now, choose  $v_h = \hat{\xi}_Z^{n+1/2}$  in (6.31),  $\tau_h = \xi_Q^{n+1/2}$  in (6.32), and  $v_h = \xi_U^{n+1/2}$  in (6.33). Adding the resulting equations, we find that

$$(6.34) \quad (\partial_t \xi_U^{n+1/2}, \xi_U^{n+1/2}) + \mathcal{A}_2(\hat{\xi}_Q^{n+1/2}, \xi_Q^{n+1/2}) + J_1(\xi_Z^{n+1/2}, \hat{\xi}_Z^{n+1/2}) + J(\hat{\xi}_U^{n+1/2}, \xi_U^{n+1/2}) \\ = -\hat{\epsilon}^{n+1/2}(\mathcal{B}(\xi_Q, \xi_Q^{n+1/2})) - \hat{\mathcal{E}}_B^{n+1/2}(\tilde{q}_h)(\xi_Q^{n+1/2}) \\ + (\partial_t \eta_U^{n+1/2}, \xi_U^{n+1/2}) + (R^n, \xi_U^{n+1/2}).$$

Note that  $\xi_U^0 = 0$ ,  $\hat{\xi}_Q^0 = 0$  and  $\xi_Q^{n+1/2} = \partial_t \hat{\xi}_Q^{n+1/2}$ , and

$$(\partial_t \xi_U^{n+1/2}, \xi_U^{n+1/2}) = \frac{1}{2k} (\|\xi_U^{n+1}\|^2 - \|\xi_U^n\|^2).$$

Other terms on the left hand side of (6.34) can be rewritten in a similar way. On substitution and then summing from  $n = 0$  to  $m$  after multiplying by  $2k$ , we arrive at

$$(6.35) \quad \|\xi_U^{m+1}\|^2 + \mathcal{A}_2(\hat{\xi}_Q^{m+1}, \hat{\xi}_Q^{m+1}) + J_1(\hat{\xi}_Z^{m+1}, \hat{\xi}_Z^{m+1}) + J(\hat{\xi}_U^{m+1}, \hat{\xi}_U^{m+1}) \\ = -2k \sum_{n=0}^m \hat{\epsilon}^{n+1/2}(\mathcal{B}(\xi_Q, \xi_Q^{n+1/2})) - 2k \sum_{n=0}^m \hat{\mathcal{E}}_B^{n+1/2}(\tilde{q}_h)(\xi_Q^{n+1/2}) \\ + 2k \sum_{n=0}^m (\partial_t \eta_U^{n+1/2}, \xi_U^{n+1/2}) + 2k \sum_{n=0}^m (R^n, \xi_U^{n+1/2}) \\ = 2(I_1^m + I_2^m + I_3^m + I_4^m).$$

For convenience, we use the following notations:  $B_r^s = B(t_s, t_r)$ , and  $\Delta_i \phi_i = (\phi_i - \phi_{i-1})/k$ . Further, let  $|||(\xi_U^n, \hat{\xi}_Q^n)|||^2 = \|\xi_U^n\|^2 + \|A^{1/2} \hat{\xi}_Q^n\|^2$  and for some  $m^* \in [0; m+1]$ , define

$$|||(\xi_U^{m^*}, \hat{\xi}_Q^{m^*})||| = \max_{0 \leq n \leq m+1} |||(\xi_U^n, \hat{\xi}_Q^n)|||.$$

For the  $I_1^m$  term, we observe that

$$\begin{aligned} \epsilon^{j+1}(B^{j+1}(\xi_Q, \xi_Q^{n+1/2})) &= k \sum_{i=0}^j B_{i+1/2}^{j+1} \xi_Q^{i+1/2} \cdot \xi_Q^{n+1/2} = k \sum_{i=0}^j B_{i+1/2}^{j+1} \partial_t \hat{\xi}_Q^{i+1/2} \cdot \xi_Q^{n+1/2} \\ &= \sum_{i=0}^j \left( B_{i+1/2}^{j+1} \hat{\xi}_Q^{i+1} - B_{i-1/2}^{j+1} \hat{\xi}_Q^i \right) \cdot \xi_Q^{n+1/2} \\ &\quad - \sum_{i=0}^j \left( B_{i+1/2}^{j+1} - B_{i-1/2}^{j+1} \right) \hat{\xi}_Q^i \cdot \xi_Q^{n+1/2} \\ &= B_{j+1/2}^{j+1} \hat{\xi}_Q^{j+1/2} \cdot \partial_t \hat{\xi}_Q^{n+1} - k \sum_{i=0}^j \left( \Delta_i(B_{i+1/2}^{j+1}) \right) \hat{\xi}_Q^i \cdot \partial_t \hat{\xi}_Q^{n+1/2}. \end{aligned}$$

Hence,

$$k^2 \sum_{n=0}^m \sum_{j=0}^n \epsilon^{j+1}(B^{j+1}(\xi_Q, \xi_Q^{n+1/2})) = k^2 \sum_{n=0}^m \Theta^{n+1} \cdot \xi_Q^{n+1/2} - k^2 \sum_{n=0}^m \Upsilon^{n+1} \cdot \xi_Q^{n+1/2},$$

where

$$\Theta^{n+1} = \sum_{j=0}^n B_{j+1/2}^{j+1} \hat{\xi}_Q^{j+1} \quad \text{and} \quad \Upsilon^{n+1} = k \sum_{j=0}^n \sum_{i=0}^j \left( \Delta_i(B_{i+1/2}^{j+1}) \right) \hat{\xi}_Q^i.$$

Next, we estimate the terms  $\Theta^{n+1}$  and  $\Upsilon^{n+1}$ . Note that

$$\begin{aligned} k^2 \sum_{n=0}^m \Theta^{n+1} \cdot \partial_t \hat{\xi}_Q^{n+1/2} &= k \sum_{n=0}^m (\Theta^{n+1} \cdot \hat{\xi}_Q^{n+1} - \Theta^n \cdot \hat{\xi}_Q^n) - k \sum_{n=0}^m (\Theta^{n+1} - \Theta^n) \cdot \hat{\xi}_Q^n \\ &= k \Theta^{m+1} \cdot \hat{\xi}_Q^{m+1} - k \sum_{n=0}^m B_{n+1/2}^{n+1} \hat{\xi}_Q^{n+1} \cdot \hat{\xi}_Q^n \\ &= k \sum_{n=0}^m B_{n+1/2}^{n+1} \hat{\xi}_Q^{n+1} \cdot \hat{\xi}_Q^{m+1} - k \sum_{n=0}^m B_{n+1/2}^{n+1} \hat{\xi}_Q^{n+1} \cdot \hat{\xi}_Q^n. \end{aligned}$$

Therefore, using the Cauchy-Schwarz inequality, we arrive at

$$\left| k^2 \sum_{n=0}^m \Theta^{n+1} \cdot \partial_t \hat{\xi}_Q^{n+1/2} \right| \leq \frac{Mk}{\alpha} \|A^{1/2} \hat{\xi}_Q^{m+1}\|^2 + C(M, \alpha) \left( k \sum_{n=0}^m \|A^{1/2} \hat{\xi}_Q^n\| \right) |||(\xi_U^{m^*}, \hat{\xi}_Q^{m^*})|||.$$

Similarly, we now obtain

$$\begin{aligned} k^2 \sum_{n=0}^m \Upsilon^{n+1} \cdot \partial_t \hat{\xi}_Q^{n+1/2} &= k \Upsilon^{m+1} \cdot \hat{\xi}_Q^{m+1} - k \sum_{n=0}^m (\Upsilon^{n+1} - \Upsilon^n) \cdot \hat{\xi}_Q^n \\ &= k^2 \left( \sum_{j=0}^m \sum_{i=0}^j \left( \Delta_i(B_{i+1/2}^{j+1}) \right) \hat{\xi}_Q^i \right) \cdot \hat{\xi}_Q^{m+1} \\ &\quad - k^2 \sum_{n=0}^m \left( \sum_{i=0}^n \left( \Delta_i(B_{i+1/2}^{n+1}) \right) \hat{\xi}_Q^i \right) \cdot \hat{\xi}_Q^n. \end{aligned}$$

Since, it is assumed that  $\|D_{t,s}B(t,s)\| \leq M$ , we deduce that

$$\left| k^2 \sum_{n=0}^m \Upsilon^{n+1} \cdot \partial_t \hat{\xi}_Q^{n+1/2} \right| \leq C(M, \alpha, T) \left( k \sum_{n=0}^m \|A^{1/2} \hat{\xi}_Q^n\| \right) |||(\xi_U^{m*}, \hat{\xi}_Q^{m*})|||.$$

Since, similar bounds can be obtained for the other terms in  $I_1^m$ , we finally conclude that

$$|I_1^m| \leq \frac{Mk}{2\alpha} \|A^{1/2} \hat{\xi}_Q^{m+1}\|^2 + C(M, \alpha, T) \left( k \sum_{n=0}^m \|A^{1/2} \hat{\xi}_Q^n\| \right) |||(\xi_U^{m*}, \hat{\xi}_Q^{m*})|||.$$

Now, with  $\Lambda^{n+1} = \sum_{j=0}^n \mathcal{E}_B^{j+1}(\tilde{\mathbf{q}}_h)$ , we observe that

$$\begin{aligned} k^2 \sum_{n=0}^m \sum_{j=0}^n \mathcal{E}_B^{j+1}(\tilde{\mathbf{q}}_h)(\xi_Q^{n+1/2}) &= k^2 \sum_{n=0}^m \Lambda^{n+1} \partial_t \hat{\xi}_Q^{n+1/2} \\ &= k \Lambda^{m+1}(\hat{\xi}_Q^{m+1}) - k \sum_{n=0}^m (\Lambda^{n+1} - \Lambda^n)(\hat{\xi}_Q^n) \\ &= k \sum_{j=0}^m \mathcal{E}_B^{j+1}(\tilde{\mathbf{q}}_h)(\hat{\xi}_Q^{m+1}) - k \sum_{n=0}^m \mathcal{E}_B^{n+1}(\tilde{\mathbf{q}}_h)(\hat{\xi}_Q^n). \end{aligned}$$

Since, the terms in  $I_2^m$  have a similar form, we deduce that

$$|I_2^m| \leq C(\alpha) k \sum_{n=0}^m (|\mathcal{E}_B^{n+1}(\tilde{\mathbf{q}}_h)|) |||(\xi_U^{m*}, \hat{\xi}_Q^{m*})|||.$$

Finally, it follows that

$$|I_3^m + I_4^m| \leq Ck \sum_{n=0}^m \left( \left\| \partial_t \eta_U^{n+1/2} \right\| + \|R^n\| \right) |||(\xi_U^{m*}, \hat{\xi}_Q^{m*})|||.$$

On substituting the above estimates in (6.35), using kickback arguments and similar arguments in previous Theorems, we arrive at

$$\begin{aligned} (1 - (M/2\alpha)k) |||(\xi_U^{m+1}, \hat{\xi}_Q^{m+1})||| &\leq Ck \sum_{n=0}^m \left( \left\| \partial_t \eta_U^{n+1/2} \right\| + \|R^n\| + \|\mathcal{E}_B^{n+1}(\tilde{\mathbf{q}}_h)\| + \|A^{1/2} \hat{\xi}_Q^n\| \right) \\ &\leq Ck \sum_{n=0}^m \left( \left\| \partial_t \eta_U^{n+1/2} \right\| + \|R^n\| + \|\mathcal{E}_B^{n+1}(\tilde{\mathbf{q}}_h)\| + |||(\xi_U^n, \hat{\xi}_Q^n)||| \right). \end{aligned}$$

Since for  $0 < k \leq k_0$ ,  $(1 - (M/2\alpha)k)$  can be made positive, then an application of the discrete Gronwall lemma yields

$$\|\xi_U^{m+1}\| + \|A^{1/2} \hat{\xi}_Q^{m+1}\| \leq Ck \sum_{n=0}^m \left( \left\| \partial_t \eta_U^{n+1/2} \right\| + \|R^n\| + \|\mathcal{E}_B^{n+1}(\tilde{\mathbf{q}}_h)\| \right).$$

The first two terms on the right hand side can be bounded as follows:

$$k \sum_{n=0}^m \left\| \partial_t \eta_U^{n+1/2} \right\| \leq \int_0^{t_{m+1}} \|\eta_{Ut}(s)\| ds,$$

and

$$k \sum_{n=0}^m \|R^n\| \leq T \max_{0 \leq n \leq m} \|R^n\| \leq kT \sum_{n=0}^m \|r^n\|.$$

Finally, by taking into account (6.29) and (6.30), we use the triangle inequality with the first estimate in Theorem 4.1 to prove the error estimates in the following theorem.

**Theorem 6.4.** *Let  $(u, \mathbf{q}, \sigma)$  be the solution of (3.1)-(3.3). Further, let  $(U^n, \mathbf{Q}^n, \mathbf{Z}^n) \in V_h \times \mathbf{W}_h \times \mathbf{W}_h$  be the solution of (6.1)-(6.4). Assume that  $U^0 = \tilde{u}_0$ ,  $\mathbf{Q}^0 = \mathbf{I}_h \nabla u_0$  and  $\mathbf{Z}^0 = \mathbf{I}_h (A \nabla u_0)$ . Then there exists a constant  $C > 0$  independent of  $h$  and  $k$ , and a constant  $k_0 > 0$  independent of  $h$ , such that for  $0 < k < k_0$  and  $m = 0, 1, \dots, N-1$*

$$\begin{aligned} \|u(t_{m+1}) - U^{m+1}\|_{L^2(\Omega)} &\leq C \frac{h^{P+D}}{p^{R+S}} \left( \|u_0\|_{H^{r+2}(\mathcal{T}_h)} + \sum_{j=0}^1 \left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^1(H^{r+2}(\mathcal{T}_h))} \right) \\ &\quad + C k^2 \sum_{j=3}^4 \int_0^{t_{m+1}} \left\| \frac{\partial^j u}{\partial t^j}(s) \right\| ds, \end{aligned}$$

where  $P, D, R$  and  $S$  as in Theorem 6.3.

**Remark 6.1.** One of the difficulties involved in the time stepping scheme applied to the present problem on hyperbolic integro-differential equation is that all the values of  $\mathbf{Q}^j$ ,  $j = 0, \dots, n-1$  have to be retained in order to compute the current value, say  $\mathbf{Q}^n$ . This, in turn, causes a great demand for data storage. One way to overcome this difficulty is to use sparse quadrature rules proposed in Sloan and Thomée [24] and analyzed for hyperbolic problems by Pani *et al.* [16]. This way, one may substantially reduce the storage requirements in the computation. Since the error analysis for the present problem with sparse quadrature rules will be quite involved, it is not possible to address it in this article.

## 7. NUMERICAL EXPERIMENTS

In this section, we present the performance of the proposed DG methods for the linear hyperbolic integro-differential equations of the form (1.1)-(1.4) with  $A = I$ ,  $B(x, t, s) = \exp(t-s)I$ ,  $\Omega = (0, 1) \times (0, 1)$  and  $T = 1$ . We divide  $\Omega$  into regular uniform closed triangles and let  $0 = t_0 < t_1 < \dots < t_N = T$  be a given partition of the time interval  $(0, T]$  with step length  $k = \frac{T}{N}$  for some positive integer  $N$ .

We first discuss the numerical procedure for the DG schemes. Let  $(\phi_i)_{i=1}^{N_h}$  be the basis functions for the finite dimensional space  $V_h$ , where  $N_h$  denotes the dimension of the space  $V_h$  and let  $(\chi_i)_{i=1}^{M_h}$  be the basis functions for the finite dimensional space  $\mathbf{W}_h$ , where  $M_h$  denotes the dimension of the space  $\mathbf{W}_h$ .

Then, we define the following matrices

$$\begin{aligned} M &= [M(i, j)]_{1 \leq i, j \leq N_h}, \quad A_1 = [A_1(i, j)]_{1 \leq i \leq N_h, 1 \leq j \leq M_h}, \\ J_1 &= [J_1(i, j)]_{1 \leq i, j \leq M_h}, \quad A = [A(i, j)]_{1 \leq i, j \leq M_h}, \quad J = [J(i, j)]_{1 \leq i, j \leq N_h}, \end{aligned}$$

and the vector

$$(7.1) \quad L = [L(i)]_{1 \leq i \leq N_h},$$

where

$$\begin{aligned} M(i, j) &= \sum_K \int_K \phi_i \phi_j dx, & A_1(i, j) &= \sum_K \int_K \phi_i \cdot \nabla \chi_j dx - \int_{\Gamma} \llbracket \chi_j \rrbracket \llbracket \phi_i \rrbracket dS, \\ J_1(i, j) &= \sum_{e \in \Gamma_I} \int_e C_{22} \llbracket \chi_i \rrbracket \llbracket \chi_j \rrbracket dS, & A(i, j)(t) &= \int_{\Omega} \chi_i \cdot \chi_j dx, \\ J(i, j) &= \sum_{e \in \Gamma} \int_e C_{11} \llbracket \phi_i \rrbracket \llbracket \phi_j \rrbracket dS, \end{aligned}$$

$$\text{and } L(i) = \sum_K \int_K f \phi_i dx.$$

Write  $U^n = \sum_{i=1}^{N_h} \alpha_i^n \phi_i$ , where  $\boldsymbol{\alpha}^n = [\alpha_1^n, \alpha_2^n, \dots, \alpha_{N_h}^n]$ ,  $\mathbf{Q}^n = \sum_{i=1}^{M_h} \beta_i^n \chi_i$ , where  $\boldsymbol{\beta}^n = [\beta_1^n, \beta_2^n, \dots, \beta_{M_h}^n]$  and  $\mathbf{Z}^n = \sum_{i=1}^{M_h} \gamma_i^n \chi_i$ , where  $\boldsymbol{\gamma}^n = [\gamma_1^n, \gamma_2^n, \dots, \gamma_{M_h}^n]$

Now using the basis functions for  $V_h$  and  $\mathbf{W}_h$ , (6.2)-(6.4) can be reduced to the following matrix form:

$$(7.2) \quad A\boldsymbol{\beta}^{n+1/2} + A_1\boldsymbol{\alpha}^{n+1/2} + J_1\boldsymbol{\gamma}^{n+1/2} = 0,$$

$$(7.3) \quad -\left(1 + \frac{k}{2} \exp(k/2)\right) A\boldsymbol{\beta}^{n+1/2} + A\boldsymbol{\gamma}^{n+1/2} = \boldsymbol{\Psi}^{n+1/2},$$

$$(7.4) \quad \left(M + \frac{k^2}{2} J\right) \boldsymbol{\alpha}^{n+1/2} - \frac{k^2}{2} A'_1 \boldsymbol{\gamma}^{n+1/2} = \boldsymbol{\Phi}^{n+1/2},$$

where

$$\boldsymbol{\Psi}^{n+1/2} = \frac{k}{2} \sum_{i=0}^{n-1} [\exp(t_{n+1} - t_{i+1/2}) + \exp(t_n - t_{i+1/2})] A\boldsymbol{\beta}^{i+1/2},$$

and

$$\boldsymbol{\Phi}^{n+1/2} = M(3\boldsymbol{\alpha}^n - \boldsymbol{\alpha}^{n-1}) + \frac{k^2}{2} [A'_1 \boldsymbol{\gamma}^{n-1/2} - J\boldsymbol{\alpha}^{n-1/2}] + k^2 L^{n+1/2}.$$

Then, the unknown vector  $[\boldsymbol{\alpha}^{n+1/2} \boldsymbol{\beta}^{n+1/2} \boldsymbol{\gamma}^{n+1/2}]$  is the solution of a linear system with a coefficient matrix

$$\mathbf{A} = \begin{bmatrix} A_1 & A & J_1 \\ 0 & -\left(1 + \frac{k}{2} \exp(k/2)\right) A & A \\ M + \frac{k^2}{2} J & 0 & -\frac{k^2}{2} A'_1 \end{bmatrix},$$

and a right hand side

$$\mathbf{b} = \begin{bmatrix} 0 \\ \boldsymbol{\Psi}^{n+1/2} \\ \boldsymbol{\Phi}^{n+1/2} \end{bmatrix}.$$

The solution will provide the values of  $U^{n+1/2}$ ,  $\mathbf{Q}^{n+1/2}$  and  $\mathbf{Z}^{n+1/2}$  for  $n = 1, \dots, N-1$ . **Convergence of  $\|u(t_n) - U^n\|$  and  $\|\boldsymbol{\sigma}(t_n) - \mathbf{Z}^n\|$ .** We show the order of convergence in the  $L^2$ -norm of the error in the flux  $\boldsymbol{\sigma}$  and in the  $L^2$ -norm of the error in the velocity  $u$ . We observe that the optimal order of convergence predicted by our theory (see Table 1) is achieved.

TABLE 2. Order of convergence of  $\|e_U(t_N)\|$ , when  $t_N = 1$ .

$C_{11} \rightarrow$	$O(1)$	$O(1)$	$O(1)$	$O(h^{-1})$	$O(h^{-1})$	$O(h^{-1})$
$C_{22} \rightarrow$	0	$O(1)$	$O(h)$	0	$O(1)$	$O(h)$
p = 1	2.1589	2.4079	2.2915	2.3508	2.2166	2.2752
p = 2	3.7140	3.4246	3.4523	3.3009	3.4130	3.1747
p = 3	4.4037	4.0778	4.5808	4.1506	3.4890	3.9036



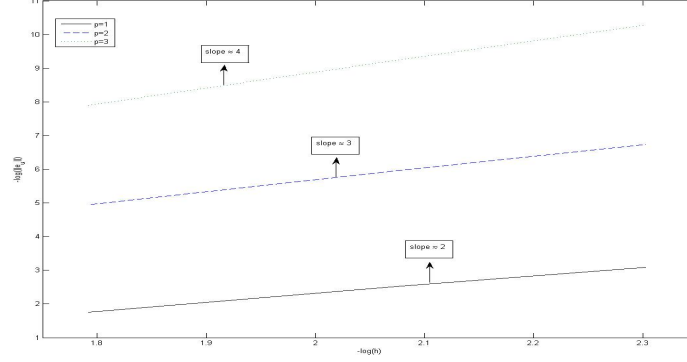


FIGURE 1. Order of convergence for  $\|e_U\|$  at  $t_N = 1$  when  $C_{11} = O(\frac{1}{h})$  and  $C_{22} = O(h)$ .

TABLE 3. Order of convergence of  $\|\mathbf{e}_Z(t_N)\|$  when  $t_N = 1$

$C_{11} \rightarrow$	$O(1)$	$O(1)$	$O(1)$	$O(h^{-1})$	$O(h^{-1})$	$O(h^{-1})$
$C_{22} \rightarrow$	0	$O(1)$	$O(h)$	0	$O(1)$	$O(h)$
p = 1	1.3815	1.1904	1.1681	1.0763	1.1414	1.0252
p = 2	2.3776	2.5265	2.3054	2.3040	2.2181	2.3379
p = 3	3.2914	3.3250	3.5694	3.5365	3.4465	3.3012

**Example:** Choose  $f$  in such a way that the exact solution is

$$u(x, y, t) = \exp(t) \sin(\pi x) \sin(\pi y).$$

We compute the order of convergence for  $e_U$  and  $\mathbf{e}_Z$  at  $t_N = 1$  for the cases  $1 \leq p \leq 3$  with different choices of stabilization parameters  $C_{11}$  and  $C_{22}$ . Tables 2 and 3 present the computed order of convergence for  $\|e_U\|$  and  $\|\mathbf{e}_Z\|$  at  $t_N = 1$ , respectively. In Figures 1 and 2, we present the convergence behavior of  $\|e_U\|$  and  $\|\mathbf{e}_Z\|$  at  $t_N = 1$ , respectively with the mesh function  $h$  and for  $1 \leq p \leq 3$  on uniform triangular meshes when  $C_{11} = O(\frac{1}{h})$  and  $C_{22} = O(h)$ . We observe that the computed order of convergence match with the predicted order of convergence.

## 8. CONCLUSION

In this paper, we have proposed and analyzed an  $hp$ -LDG method for a hyperbolic type integro-differential equation. Compared to the elliptic case [5], [18], we have, in this article, established similar  $hp$ -error estimates for the semidiscrete scheme after suitably modifying the numerical fluxes. Due to the presence of integral term, an introduction of an expanded mixed Ritz-Volterra projection helps to achieve optimal estimates. Further, we have applied a second order implicit method to the semidiscrete scheme to derive a completely discrete scheme and have derived optimal error estimates. Finally, we have also discussed some numerical results.

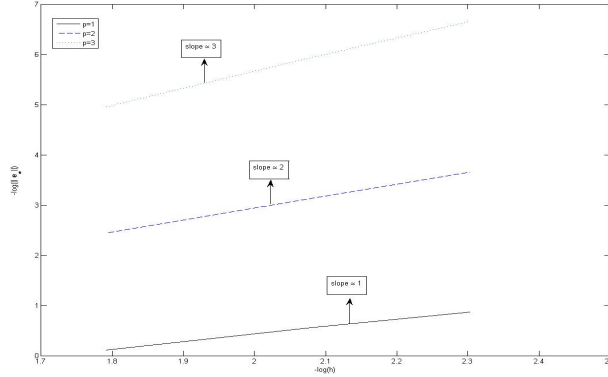


FIGURE 2. Order of convergence for  $\|\mathbf{e}_z\|$  at  $t_N = 1$  when  $C_{11} = O(\frac{1}{h})$  and  $C_{22} = O(h)$ .

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